# $4 \times 4$ MATRICES IN DIRAC PARAMETRIZATION: INVERSION PROBLEM AND DETERMINANT

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Parametrization of complex $4\times4$ – matrices $G$ in terms of Dirac tensor parameters $(A, B, A_l, B_l, F_{kl})$ or equivalent four complex 4-vectors $(k, m, n, l)$ is investigated. In the given parametrization, the problem of inverting any $4 \times 4$ matrix $G$ is solved. Expression for determinant of any matrix $G$ is found: det $G = F(k, m, n, l)$.

## 1 Introduction

In the context of group theory, the Dirac matrices-based approach was widely used in physical context: Macfarlane [1]-[2], Hermann [4], Kilhberg [5]. Mack-Todorov [7], ten Kate [3], Mack – Salam [6], arut - Bohm [8], Mack [10], Barut – Bracken [9]-[11]. Barut – Zeni – Laufer [12], Gsponer [13], Ramakrishna – Costa [14]. However, usually they exploit only general properties of Dirac basis to parameterize $4 \times 4$ matrices. In the present paper we consider three problems linked to Dirac matrices based approach:

1) Dirac matrix basis and multiplication in $GL(4.C)$
2) Inverse matrix $G^{-1}$
3) Determinant $\mid G \mid$ in the Dirac parameters

the problems are rather labarous technically, but result seem to be important for applications.

## 2 Dirac matrix basis and multiplication in $GL(4.C)$

Any complex matrix $G \in GL(4.C)$ can be resolved in terms of 16 Dirac matrices:

$$G = A\, I + iB\, \gamma^5 + iA_l\, \gamma^l + B_l\, \gamma^l \gamma^5 + F_{mn}\, \sigma_{mn}\, , \tag{1}$$

the notation is used

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}, \qquad g^{ab} = \text{diag}(+1, -1, -1, -1)\, ,$$
$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\, , \qquad \sigma^{ab} = \frac{1}{4}\, (\gamma^a\gamma^b - \gamma^b\gamma^a)\, . \tag{2}$$

16 coefficients may be taken as independent parameters in $GL(4.C)$. To establish the composition law for parameters one should multiply any two matrices of the type (1) and the result

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obtained is to be decomposed again in terms of Dirac matrices:

$$
\begin{aligned}
G' &= A'\; I + iB'\; \gamma^5 + iA'_k\; \gamma^k + B'_k\; \gamma^k \gamma^5 + F'_{cd}\; \sigma^{cd}\;, \\
G &= A\; I + iB\; \gamma^5 + iA_l\; \gamma^l + B_l\; \gamma^l \gamma^5 + F_{mn}\; \sigma^{mn}\;, \\
G'' = G'G &= A''\; I + iB''\; \gamma^5 + iA''_l\; \gamma^l + B''_l\; \gamma^l \gamma^5 + F''_{mn}\; \sigma_{mn}\;.
\end{aligned}
$$

$$
\begin{aligned}
G'' = I\; A'A + iA'B\; \gamma^5 + iA'A_k\; \gamma^k + A'B_k\; \gamma^k\gamma^5 + A'F_{cd}\; \sigma^{cd} \\
+iB'A\; \gamma^5 - B'B\; I + B'A_k\; \gamma^k\gamma^5 - iB'B_k\; \gamma^k + iB'F_{cd}\; \sigma^{cd}\gamma^5 \\
+iA'_kA\; \gamma^k - A'_kB\; \gamma^k\gamma^5 - A'_kA_l\; \gamma^k\gamma^l + iA'_kB_l\; \gamma^k\gamma^l\gamma^5 + iA'_lF_{cd}\; \gamma^l\sigma^{cd} \\
+B'_kA\; \gamma^k\gamma^5 + iB'_kB\; \gamma^k - iB'_kA_l\; \gamma^k\gamma^l\gamma^5 - B'_kB_l\; \gamma^k\gamma^l + B'_lF_{cd}\; \gamma^l\sigma^{cd}\gamma^5 \\
+F'_{mn}A\; \sigma^{mn} + iF'_{mn}B\; \sigma^{mn}\gamma^5 + iF'_{mn}A_k\; \sigma^{mn}\gamma^k \\
+F'_{mn}B_k\; \sigma^{mn}\gamma^k\gamma^5 + F'_{mn}F_{cd}\; \sigma^{mn}\sigma^{cd}\;.
\end{aligned} \tag{3}
$$

We need some subsidiary relations, they are well known but for more completeness let us specify some details. The main formula, base for calculation with Dirac matrices, look as follows

$$
\begin{gathered}
\gamma^a\gamma^b\gamma^c = \gamma^a g^{bc} - \gamma^b g^{ac} + \gamma^c g^{ab} + i\gamma^5\; \epsilon^{abcd}\; \gamma_d\;, \\
\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\; \epsilon_{abcd}\; \gamma^a\gamma^b\gamma^c\gamma^d\;, \qquad \epsilon^{0123} = +1\;.
\end{gathered} \tag{4}
$$

There are several evident formulas:

$$
\gamma^a\gamma^b = I\; g^{ab} + 2\sigma^{ab}\;;
$$

also

$$
\sigma^{ab}\; \gamma^5 = -\frac{i}{2}\; \epsilon^{abkl}\; \sigma_{kl}\;, \qquad \gamma^5\; \sigma^{ab} = -\frac{i}{2}\; \epsilon^{abkl}\; \sigma_{kl}\;; \tag{5}
$$

also

$$
\begin{aligned}
\gamma^a\gamma^b\; \gamma^5 &= (g^{ab} + 2\sigma^{ab})\; \gamma^5 = g^{ab}\; \gamma^5 - i\; \epsilon^{abkl}\; \sigma_{kl}\;, \\
\gamma^5\; \gamma^a\gamma^b &= \gamma^5\; (g^{ab} + 2\sigma^{ab})\; = g^{ab}\; \gamma^5 - i\; \epsilon^{abkl}\; \sigma_{kl}\;.
\end{aligned} \tag{6}
$$

From identity

$$
\gamma^l\sigma^{cd} = \frac{1}{4}\gamma^l\; (\gamma^c\gamma^d - \gamma^d\gamma^c) =
$$

$$
= \frac{1}{4}\; \Big[\; \gamma^l g^{cd} - \gamma^c g^{ld} + \gamma^d g^{lc} + i\gamma^5\epsilon^{lcds}\gamma_s - \gamma^l g^{dc} + \gamma^d g^{lc} - \gamma^c g^{ld} - i\gamma^5\epsilon^{ldcs}\gamma_s\; \Big]\,,
$$

it follows

$$
\gamma^l\sigma^{cd} = \frac{1}{2}\; \Big[\; g^{lc}\gamma^d - g^{ld}\gamma^c + i\gamma^5\epsilon^{lcds}\gamma_s\; \Big]\,, \tag{7}
$$

in the same manner

$$
\sigma^{mn}\gamma^k = \frac{1}{2}\; \Big[\; \gamma^m g^{nk} - \gamma^n g^{mk} + i\gamma^5\epsilon^{mnks}\gamma_s\; \Big]\,. \tag{8}
$$

There are two similar formulas with involved $\gamma^5$:

$$\gamma^l \sigma^{cd} \gamma^5 = \frac{1}{2} \left[ g^{lc} \gamma^d \gamma^5 - g^{ld} \gamma^c \gamma^5 - i \epsilon^{lcds} \gamma_s \right] ,$$
$$\sigma^{mn} \gamma^k \gamma^5 = \frac{1}{2} \left[ \gamma^m \gamma^5 g^{nk} - \gamma^n \gamma^5 g^{mk} - i \epsilon^{mnks} \gamma_s \right] . \tag{9}$$

Finally, we need one other combination

$$\sigma^{mn} \sigma^{cd} = \frac{1}{4} \gamma^m (\gamma^n \sigma^{cd}) - \frac{1}{4} \gamma^n (\gamma^m \sigma^{cd})$$
$$= \frac{1}{8} \left[ \gamma^m ( g^{nc} \gamma^d - g^{nd} \gamma^c + i \gamma^5 \epsilon^{ncds} \gamma_s ) - \gamma^n ( g^{mc} \gamma^d - g^{md} \gamma^c + i \gamma^5 \epsilon^{mcds} \gamma_s ) \right]$$
$$= \frac{1}{8} \left[ (g^{nc} \gamma^m \gamma^d - g^{nd} \gamma^m \gamma^c) - (g^{mc} \gamma^n \gamma^d - g^{md} \gamma^n \gamma^c) - (i \epsilon^{ncd}{}_s \gamma^m \gamma^s \gamma^5 - i \epsilon^{mcd}{}_s \gamma^n \gamma^s \gamma^5) \right] .$$

which gives

$$\sigma^{mn} \sigma^{cd} = \frac{1}{8} \{ [ g^{nc} (g^{md} + 2\sigma^{md}) - g^{nd} (g^{mc} + 2\sigma^{mc}) ]$$
$$-[ (g^{mc} (g^{nd} + 2\sigma^{nd}) - g^{md} (g^{nc} + 2\sigma^{nc}) ]$$
$$-[ i \epsilon^{ncd}{}_s (g^{ms} \gamma^5 - i \epsilon^{mskl} \sigma_{kl}) - i \epsilon^{mcd}{}_s (g^{ns} \gamma^5 - i \epsilon^{nskl} \sigma_{kl}) ] \}$$
$$= \frac{1}{8} \{ [ (g^{nc} g^{md} - g^{nd} g^{mc}) - (g^{mc} g^{nd} - g^{md} g^{nc}) ]$$
$$+2[ (g^{nc} \sigma^{md} - g^{nd} \sigma^{mc}) - (g^{mc} \sigma^{nd} - g^{md} \sigma^{nc}) ] + 2i \epsilon^{mncd} \gamma^5$$
$$-( \epsilon^{ncd}{}_s \epsilon^{mkls} \sigma_{kl} - \epsilon^{mcd}{}_s \epsilon^{nkls} \sigma_{kl} ) \}.$$

Further, using the known identity

$$\epsilon^{ncd}{}_s \epsilon^{mkls} \sigma_{kl} = - \begin{vmatrix} g^{mn} & g^{mc} & g^{md} \\ g^{kn} & g^{kc} & g^{kd} \\ g^{ln} & g^{lc} & g^{ld} \end{vmatrix} \sigma_{kl} ,$$

after simple calculation we get

$$\sigma^{mn} \sigma^{cd} = \frac{1}{8} [ (g^{nc} g^{md} - g^{nd} g^{mc}) - (g^{mc} g^{nd} - g^{md} g^{nc}) ]$$
$$+ \frac{i}{4} \epsilon^{mncd} \gamma^5 + \frac{1}{2} [ (g^{nc} \sigma^{md} - g^{nd} \sigma^{mc}) - (g^{mc} \sigma^{nd} - g^{md} \sigma^{nc}) ] . \tag{10}$$

From (3) we arrive at

$$G'' = G' G = A'' I + i B'' \gamma^5 + i A''_l \gamma^l + B''_l \gamma^l \gamma^5 + F''_{mn} \sigma_{mn}$$
$$= A' A I + i A' B \gamma^5 + i A' A_l \gamma^l + A' B_l \gamma^l \gamma^5 + A' F_{kl} \sigma^{kl}$$
$$+ i B' A \gamma^5 - B' B I + B' A_l \gamma^l \gamma^5 - i B' B_l \gamma^l + i B' F_{cd} (-i/2) \epsilon^{cdkl} \sigma_{kl}$$
$$+ i A'_l A \gamma^l - A'_l B \gamma^l \gamma^5 - A'_l A_k (g^{lk} + 2\sigma^{lk}) + i A'_l B_k (g^{lk} \gamma^5 - i \epsilon^{lkmn} \sigma_{mn})$$

$$+ iA'_l F_{cd}\,\frac{1}{2}\,[\,(g^{lc}\gamma^d - g^{ld}\gamma^c) + i\,\gamma^5\epsilon^{lcds}\gamma_s\,] + B'_l A\,\gamma^l\gamma^5 + iB'_l B\,\gamma^l$$
$$-iB'_l A_k\,(g^{lk}\gamma^5 - i\,\epsilon^{lkmn}\,\sigma_{mn}) - B'_l B_k\,(g^{lk} + 2\sigma^{lk})$$
$$+B'_l F_{cd}\,\frac{1}{2}\,[\,(g^{lc}\gamma^d\gamma^5 - g^{ld}\gamma^c\gamma^5) - i\,\epsilon^{lcds}\,\gamma_s\,] + F'_{mn}A\,\sigma^{mn}$$
$$+iF'_{mn}B\,(-i/2)\,\epsilon^{mnkl}\,\sigma_{kl} + iF'_{mn}A_k\,\frac{1}{2}\,[\,(\gamma^m g^{nk} - \gamma^n g^{mk}) + i\gamma^5\,\epsilon^{mnks}\,\gamma_s\,]$$
$$+F'_{mn}B_k\,\frac{1}{2}\,[\,(\gamma^m\gamma^5 g^{nk} - \gamma^n\gamma^5 g^{mk}) - i\,\epsilon^{mnks}\,\gamma_s\,]$$
$$+F'_{mn}F_{cd}\,\{\frac{1}{8}\,[\,(g^{nc}g^{md} - g^{nd}g^{mc}) - (g^{mc}g^{nd} - g^{md}g^{nc})\,]$$
$$+\frac{i}{4}\,\epsilon^{mncd}\,\gamma^5 + \frac{1}{2}\,[\,(g^{nc}\sigma^{md} - g^{nd}\sigma^{mc}) - (g^{mc}\sigma^{nd} - g^{md}\sigma^{nc})\,]\}\;. \qquad (11)$$

In the first place, expression for two scalars are produced:

$$A'' = A'\,A - B'\,B - A'_l\,A^l - B'_l\,B^l - \frac{1}{2}\,F'_{kl}\,F^{kl}\;,$$
$$B'' = A'\,B + B'\,A + A'_l\,B^l - B'_l\,A^l + \frac{1}{4}\,F'_{mn}\,F_{cd}\,\epsilon^{mncd}\;. \qquad (12)$$

Now, from

$$iA''_l\,\gamma^l = iA'A_l\,\gamma^l - iB'B_l\,\gamma^l + iA'_l A\,\gamma^l + iA'_l F_{cd}\,\frac{1}{2}\,(g^{lc}\gamma^d - g^{ld}\gamma^c) + iB'_l B\,\gamma^l$$
$$-iB'_l F_{cd}\,\frac{1}{2}\,\epsilon^{lcds}\,\gamma_s + iF'_{mn}A_k\,\frac{1}{2}\,(\gamma^m g^{nk} - \gamma^n g^{mk}) - \frac{i}{2}\,F'_{mn}B_k\,\epsilon^{mnks}\,\gamma_s\;,$$

and

$$\gamma^l\gamma^5\,B''_l = A'B_l\,\gamma^l\gamma^5 + B'A_l\,\gamma^l\gamma^5 - A'_l B\,\gamma^l\gamma^5 + A'_l F_{cd}\,\frac{1}{2}\,\epsilon^{lcds}\,\gamma_s\gamma^5 + B'_l A\,\gamma^l\gamma^5$$
$$+B'_l F_{cd}\,\frac{1}{2}\,(g^{lc}g^d\gamma^5 - g^{ld}g^c\gamma^5) + \frac{1}{2}\,F'_{mn}A_k\,\epsilon^{mnks}\,\gamma_s\gamma^5 + \frac{1}{2}\,F'_{mn}B_k\,(\gamma^m\gamma^5 g^{nk} - \gamma^n\gamma^5 g^{mk})\;.$$

it follow expressions for $A''_l$ and $B''_l$:

$$A''_l = A'\,A_l - B'\,B_l + A'_l\,A + B'_l\,B + A'^k\,F_{kl}$$
$$+F'_{lk}\,A^k + \frac{1}{2}\,B'_k\,F_{mn}\,\epsilon_l^{\;\;kmn} + \frac{1}{2}\,F'_{mn}\,B_k\,\epsilon_l^{\;\;mnk}\;; \qquad (13)$$
$$B''_l = A'\,B_l + B'\,A_l - A'_l\,B + B'_l\,A + B'^k\,F_{kl}$$
$$+F'_{lk}\,B^k + \frac{1}{2}\,A'_k\,F_{mn}\,\epsilon^{kmn}{}_l + \frac{1}{2}\,F'_{mn}\,A_k\,\epsilon^{mnk}{}_l\;. \qquad (14)$$

Finally, because

$$\sigma^{mn}\,F_{mn} = A'F_{mn}\sigma^{mn} + \frac{1}{2}\,B'F_{cd}\,\epsilon^{cdmn}\,\sigma_{mn} - 2A'_m A_n\,\sigma^{mn} + A'_l B_k\,\epsilon^{lkmn}\,\sigma^{mn}$$
$$-B'_l A_k\,\epsilon^{lkmn}\,\sigma_{mn} - 2B'_m B_n\,\sigma^{mn} + F'_{mn}A\,\sigma^{mn} + \frac{1}{2}\,F'_{kl}B\,\epsilon^{klmn}\,\sigma_{mn}$$
$$+\frac{1}{2}\,F'_{mn}F_{cd}\,[\,(g^{nc}\sigma^{md} - g^{nd}\sigma^{mc}) - (g^{mc}\sigma^{nd} - g^{md}\sigma^{nc})\,]\;,$$

the tensor quantity $F''_{mn}$ is

$$F''_{mn} = A' \; F_{mn} + F'_{mn} \; A - (A'_m \; A_n - A'_n \; A_m) - (B'_m \; B_n - B'_n \; B_m)$$
$$+A'_l \; B_k \; \epsilon^{lkmn} - B'_l \; A_k \; \epsilon^{lkmn} \; + \frac{1}{2} \; B' \; F_{kl} \; \epsilon^{kl}{}_{mn} + \frac{1}{2} \; F'_{kl} \; B \; \epsilon^{kl}{}_{mn}$$
$$+(F'_{mk} \; F^{k}{}_{n} - F'_{nk} \; F^{k}{}_{m}) \; . \tag{15}$$

Thus, multiplication law for the group $GL(4.C)$, and all its sub-groups are described by one the same formula:

$$G'G = A'' \; I + iB'' \; \gamma^5 + iA''_l \; \gamma^l + B''_l \; \gamma^l \gamma^5 + F''_{mn} \; \sigma_{mn} \; ,$$

$$A'' = A' \; A - B' \; B - A'_l \; A^l - B'_l \; B^l - \frac{1}{2} \; F'_{kl} \; F^{kl} \; ,$$
$$B'' = A' \; B + B' \; A + A'_l \; B^l - B'_l \; A^l + \frac{1}{4} \; F'_{mn} \; F_{cd} \; \epsilon^{mncd} \; ,$$
$$A''_l = A' \; A_l - B' \; B_l + A'_l \; A + B'_l \; B + A'^k F_{kl}$$
$$+F'_{lk} A^k + \frac{1}{2} \; B'_k \; F_{mn} \; \epsilon_l{}^{kmn} + \frac{1}{2} \; F'_{mn} \; B_k \; \epsilon_l{}^{mnk} \; ,$$
$$B''_l = A' \; B_l + B' \; A_l - A'_l \; B + B'_l \; A + B'^k \; F_{kl}$$
$$+F'_{lk} \; B^k + \frac{1}{2} \; A'_k \; F_{mn} \; \epsilon^{kmn}{}_l + \frac{1}{2} \; F'_{mn} \; A_k \; \epsilon^{mnk}{}_l \; ,$$
$$F''_{mn} = A' \; F_{mn} + F'_{mn} \; A - (A'_m \; A_n - A'_n \; A_m) - (B'_m \; B_n - B'_n \; B_m)$$
$$+A'_l \; B_k \; \epsilon^{lkmn} - B'_l \; A_k \; \epsilon^{lkmn} \; + \frac{1}{2} \; B' \; F_{kl} \; \epsilon^{kl}{}_{mn} + \frac{1}{2} \; F'_{kl} \; B \; \epsilon^{kl}{}_{mn}$$
$$+(F'_{mk} \; F^{k}{}_{n} - F'_{nk} \; F^{k}{}_{m}) \; . \tag{16}$$

## 3 Inverse matrix $G^{-1}$

Let a matrix $G$ is given by

$$G = \left| \begin{array}{cccc} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ & & & \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{array} \right| , \tag{17}$$

For inverse matrix we have general expression

$$G^{-1} = \mid G \mid^{-1} \left| \begin{array}{cccc} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{array} \right| . \tag{18}$$

Let us find $(k_0)^{-1}$ and $(k_3)^{-1}$:

$$\frac{A_{11} + A_{22}}{2 \; \det G} = (k_0)^{-1} \; , \qquad \frac{A_{11} - A_{22}}{2 \; \det G} = (k_3)^{-1} \; . \tag{19}$$

Cofactor $A_{11}$ is

$$A_{11} = \begin{vmatrix} k_0 - k_3 & -n_1 - in_2 & n_0 + n_3 \\ -l_1 + il_2 & m_0 - m_3 & -m_1 + im_2 \\ -l_0 + l_3 & -m_1 - im_2 & m_0 + m_3 \end{vmatrix} =$$

$$\begin{aligned} = (k_0 - k_3)\ (mm) + (n_0 + n_3)(l_1 - il_2)(m_1 + im_2) \\ -(l_0 - l_3)(n_1 + in_2)(m_1 - im_2) \\ -(m_0 + m_3)(l_1 - il_2)(n_1 + in_2) \\ +(l_0 - l_3)(m_0 - m_3)(n_0 + n_3)\ , \end{aligned} \tag{20}$$

Cofactor $A_{22}$ is

$$A_{22} = \begin{vmatrix} k_0 + k_3 & n_0 - n_3 & -n_1 + in_2 \\ -l_0 - l_3 & m_0 - m_3 & -m_1 + im_2 \\ -l_1 - il_2 & -m_1 - im_2 & m_0 + m_3 \end{vmatrix} =$$

$$\begin{aligned} = (k_0 + k_3)\ (mm) + (n_0 - n_3)(l_1 + il_2)(m_1 - im_2) \\ -(l_0 + l_3)(n_1 - in_2)(m_1 + im_2) \\ -(m_0 - m_3)(l_1 + il_2)(n_1 - in_2) \\ +(l_0 + l_3)(m_0 + m_3)(n_0 - n_3)\ . \end{aligned} \tag{21}$$

With the use of identities

$$\frac{1}{2}\ [(n_0 + n_3)(l_1 - il_2)(m_1 + im_2) + (n_0 - n_3)(l_1 + il_2)(m_1 - im_2)]$$
$$= n_0 l_1 m_1 + n_0 l_2 m_2 + in_3 l_1 m_2 - in_3 l_2 m_1\ ,$$
$$\frac{1}{2}\ [(n_0 + n_3)(l_1 - il_2)(m_1 + im_2) - (n_0 - n_3)(l_1 + il_2)(m_1 - im_2)]$$
$$= in_0 l_1 m_2 - in_0 l_2 m_1 + n_3 l_1 m_1 + n_3 l_2 m_2\ ,$$

$$-\frac{1}{2}\ [\ (l_0 - l_3)(n_1 + in_2)(m_1 - im_2) + (l_0 + l_3)(n_1 - in_2)(m_1 + im_2)\ ]$$
$$= -l_0 n_1 m_1 - l_0 n_2 m_2 - il_3 n_1 m_2 + il_3 n_2 m_1\ ,$$
$$-\frac{1}{2}\ [\ (l_0 - l_3)(n_1 + in_2)(m_1 - im_2) - (l_0 + l_3)(n_1 - in_2)(m_1 + im_2)\ ]$$
$$= il_0 n_1 m_2 - il_0 n_2 m_1 + l_3 n_1 m_1 + l_3 n_2 m_2\ .$$

$$-\frac{1}{2}\ [\ (m_0 + m_3)(l_1 - il_2)(n_1 + in_2) + (m_0 - m_3)(l_1 + il_2)(n_1 - in_2)\ ]$$
$$= -m_0 l_1 n_1 - m_0 l_2 n_2 - im_3 l_1 n_2 + im_3 l_2 n_1\ ,$$
$$-\frac{1}{2}\ [\ (m_0 + m_3)(l_1 - il_2)(n_1 + in_2) - (m_0 - m_3)(l_1 + il_2)(n_1 - in_2)\ ]$$
$$= -im_0 l_1 n_2 + im_0 l_2 n_1 - m_3 l_1 n_1 - m_3 l_2 n_2\ ,$$

$$\frac{1}{2}\ [\ (l_0 - l_3)(m_0 - m_3)(n_0 + n_3) + (l_0 + l_3)(m_0 + m_3)(n_0 - n_3)\ ]$$
$$= l_0 m_0 n_0 - l_0 m_3 n_3 - l_3 m_0 n_3 + l_3 m_3 n_0\ ,$$
$$\frac{1}{2}\ [\ (l_0 - l_3)(m_0 - m_3)(n_0 + n_3) - (l_0 + l_3)(m_0 + m_3)(n_0 - n_3)\ ]$$
$$= l_0 m_0 n_3 - l_0 m_3 n_0 - l_3 m_0 n_0 + l_3 m_3 n_3\ ;,$$

we find $(k_0)^{-1}$ and $(k_3)^{-1}$:

$$
\begin{aligned}
(k_0)^{-1} &= +k_0\ (mm) \\
&+n_0 l_1 m_1 + n_0 l_2 m_2 + i n_3 l_1 m_2 - i n_3 l_2 m_1 \\
&-l_0 n_1 m_1 - l_0 n_2 m_2 - i l_3 n_1 m_2 + i l_3 n_2 m_1 \\
&-m_0 l_1 n_1 - m_0 l_2 n_2 - i m_3 l_1 n_2 + i m_3 l_2 n_1 \\
&+l_0 m_0 n_0 - l_0 m_3 n_3 - l_3 m_0 n_3 + l_3 m_3 n_0\ ,
\end{aligned}
$$

$$
\begin{aligned}
(k_3)^{-1} &= -k_3\ (mm) \\
&+i n_0 l_1 m_2 - i n_0 l_2 m_1 + n_3 l_1 m_1 n_3 l_2 m_2 + \\
&+i l_0 n_1 m_2 - i l_0 n_2 m_1 + l_3 n_1 m_1 + l_3 n_2 m_2 \\
&-i m_0 l_1 n_2 + i m_0 l_2 n_1 - m_3 l_1 n_1 - m_3 l_2 n_2 \\
&+l_0 m_0 n_3 - l_0 m_3 n_0 - l_3 m_0 n_0 + l_3 m_3 n_3\ .
\end{aligned}
$$

From this, after identical transformations, we arrive at (for brevity the factor $\mid G \mid^{-1}$ is omitted)

$$
(k_0)^{-1} = k_0\ (mm) + m_0\ (ln) + l_0\ (nm) - n_0 (lm)\ + i\, \mathbf{l}\, (\mathbf{m} \times \mathbf{n})\ , \tag{22}
$$

where

$$
i\, \mathbf{l}\, (\mathbf{m} \times \mathbf{n}) = i\ [\ l_1 (m_2 n_3 - m_3 n_2) + l_2 (m_3 n_1 - m_1 n_3) + l_3 (m_1 n_2 - m_2 n_1)\ ]\ ;
$$

and

$$
\begin{gathered}
(k_3)^{-1} = -k_3\ (mm) - m_3\ (ln) - l_3\ (nm) + n_3 (lm) \\
+2\ [\ \mathbf{l} \times (\mathbf{n} \times \mathbf{m})\ ]_3\ + i\ [\ m_0 (\mathbf{n} \times \mathbf{l})_3 + l_0 (\mathbf{n} \times \mathbf{m})_3 + n_0 (\mathbf{l} \times \mathbf{m})_3\ ]\ ,
\end{gathered} \tag{23}
$$

where

$$
\begin{aligned}
2\ [\ \mathbf{l} \times (\mathbf{n} \times \mathbf{m})\ ]_3 &= 2\ [\ l_1\ (n_3 m_1 - n_1 m_3) - l_2 (n_2 m_3 - n_3 m_2)\ ]\ , \\
i\ [\ m_0 (\mathbf{n} \times \mathbf{l})_3 &+ l_0 (\mathbf{n} \times \mathbf{m})_3 + n_0 (\mathbf{l} \times \mathbf{m})_3\ ] \\
&= i\ [\ m_0 (n_1 l_2 - n_2 l_1) + l_0 (n_1 m_2 - n_2 m_1) + n_0 (l_1 m_2 - l_2 m_1)\ ]\ .
\end{aligned}
$$

Now, let us find $(k_1)^{-1}$ and $(k_1)^{-1}$:

$$
k_1 = \frac{1}{2 \mid G \mid}\ (A_{12} + A_{21})\ , \qquad i k_2 = \frac{1}{2 \mid G \mid}\ (A_{12} - A_{21})\ . \tag{24}
$$

Cofactor $A_{12}$ is

$$
\begin{aligned}
A_{12} = (-1) &\begin{vmatrix} k_1 + i k_2 & -n_1 - i n_2 & n_0 + n_3 \\ -l_0 - l_3 & m_0 - m_3 & -m_1 + i m_2 \\ -l_1 - i l_2 & -m_1 - i m_2 & m_0 + m_3 \end{vmatrix} \\
= (-1) \{\ (k_1 + i k_2)\ (mm) &+ (l_0 + l_3)(n_0 + n_3)(m_1 + i m_2) \\
&+ (m_0 - m_3)(n_0 + n_3)(l_1 + i l_2) \\
&- (m_0 + m_3)(l_0 + l_3)(n_1 + i n_2) \\
&- (l_1 + i l_2)(n_1 + i n_2)(m_1 - i m_2)\ \}\ .
\end{aligned} \tag{25}
$$

Cofactor $A_{21}$ is

$$
\begin{aligned}
A_{21} = (-1) & \begin{vmatrix} k_1 - ik_2 & n_0 - n_3 & -n_1 + in_2 \\ -l_1 + il_2 & m_0 - m_3 & -m_1 + im_2 \\ -l_0 + l_3 & -m_1 - im_2 & m_0 + m_3 \end{vmatrix} \\
= (-1)\{ & (k_1 - ik_2)\ (mm) + (l_0 - l_3)(n_0 - n_3)(m_1 - im_2) \\
& +(m_0 - m_3)(n_0 - n_3)(l_1 - il_2) \\
& -(m_0 - m_3)(l_0 - l_3)(n_1 - in_2) \\
& -(l_1 - il_2)(n_1 - in_2)(m_1 + im_2) \ \} \ .
\end{aligned} \tag{26}
$$

With the use of identities:

$$
\begin{aligned}
& \frac{1}{2} \left[ \ (l_0 + l_3)(n_0 + n_3)(m_1 + im_2) + (l_0 - l_3)(n_0 - n_3)(m_1 - im_2) \ \right] \\
& \qquad = l_0 n_0 m_1 + l_3 n_3 m_1 + il_0 n_3 m_2 + il_3 n_0 m_2 \ , \\
& \frac{1}{2} \left[ \ (l_0 + l_3)(n_0 + n_3)(m_1 + im_2) - (l_0 - l_3)(n_0 - n_3)(m_1 - im_2) \ \right] \\
& \qquad = l_0 n_3 m_1 + l_3 n_0 m_1 + il_0 n_0 m_2 + il_3 n_3 m_2 \ ,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2} \left[ \ (m_0 - m_3)(n_0 + n_3)(l_1 + il_2) + (m_0 - m_3)(n_0 - n_3)(l_1 - il_2) \ \right] \\
& \qquad = n_0 m_0 l_1 - m_3 n_3 l_1 + im_0 n_3 l_2 - in_0 m_3 l_2 \ , \\
& \frac{1}{2} \left[ \ (m_0 - m_3)(n_0 + n_3)(l_1 + il_2) + (m_0 - m_3)(n_0 - n_3)(l_1 - il_2) \ \right] \\
& \qquad = m_0 n_3 l_1 - m_3 n_0 l_1 + im_0 n_0 l_2 - im_3 n_3 l_2 \ ,
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2} \left[ \ (m_0 + m_3)(l_0 + l_3)(n_1 + in_2) + (m_0 - m_3)(l_0 - l_3)(n_1 - in_2) \ \right] \\
& \qquad = -m_0 l_0 n_1 - m_3 l_3 n_1 - im_0 l_3 n_2 - im_3 l_0 n_2 \ , \\
& -\frac{1}{2} \left[ \ (m_0 + m_3)(l_0 + l_3)(n_1 + in_2) - (m_0 - m_3)(l_0 - l_3)(n_1 - in_2) \ \right] \\
& \qquad = -m_0 l_3 n_1 - m_3 l_0 n_1 - im_0 l_0 n_2 - im_3 l_3 n_2 \ ,
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2} \left[ \ (l_1 + il_2)(n_1 + in_2)(m_1 - im_2) + (l_1 - il_2)(n_1 - in_2)(m_1 + im_2) \ \right] \\
& \qquad = -l_1 n_1 m_1 - l_1 n_2 m_2 - l_2 m_2 n_1 + l_2 n_2 m_1 \ , \\
& -\frac{1}{2} \left[ \ (l_1 + il_2)(n_1 + in_2)(m_1 - im_2) - (l_1 - il_2)(n_1 - in_2)(m_1 + im_2) \ \right] \\
& \qquad = +il_1 n_1 m_2 - il_1 m_1 n_2 - il_2 m_1 n_1 - il_2 n_2 m_2 \ ,
\end{aligned}
$$

we find $(k_1)^{-1}$ and $(k_2)^{-1}$:

$$
\begin{aligned}
(k_1)^{-1} = (-1)\ \{ k_1\ (mm) \\
+l_0 n_0 m_1 + l_3 n_3 m_1 + il_0 n_3 m_2 + il_3 n_0 m_2 \\
+n_0 m_0 l_1 - m_3 n_3 l_1 + im_0 n_3 l_2 - in_0 m_3 l_2 - \\
-m_0 l_0 n_1 - m_3 l_3 n_1 - im_0 l_3 n_2 - im_3 l_0 n_2 \\
-l_1 n_1 m_1 - l_1 n_2 m_2 - l_2 m_2 n_1 + l_2 n_2 m_1 \ \},
\end{aligned}
$$

$$
\begin{aligned}
i(k_2)^{-1} = (-1)\ \{ik_2\ (mm) \\
+l_0 n_3 m_1 + l_3 n_0 m_1 + i l_0 n_0 m_2 + i l_3 n_3 m_2 \\
+m_0 n_3 l_1 - m_3 n_0 l_1 + i m_0 n_0 l_2 - i m_3 n_3 l_2 \\
-m_0 l_3 n_1 - m_3 l_0 n_1 - i m_0 l_0 n_2 - i m_3 l_3 n_2 \\
+i l_1 n_1 m_2 - i l_1 m_1 n_2 - i l_2 m_1 n_1 - i l_2 n_2 m_2\ .
\end{aligned}
$$

From this, after identical transformations, we arrive at

$$
\begin{aligned}
(k_1)^{-1} = -k_1\ (mm) - m_1\ (ln) - l_1\ (nm) + n_1(lm) + 2\ [\ \mathbf{l} \times (\mathbf{n} \times \mathbf{m})\ ]_1 \\
+i\ [\ m_0(\mathbf{n} \times \mathbf{l})_1 + l_0(\mathbf{n} \times \mathbf{m})_1 + n_0(\mathbf{l} \times \mathbf{m})_1\ ]\ ,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
(k_2)^{-1} = -k_2\ (mm) - m_2\ (ln) - l_2\ (nm) n_2(lm) + 2\ [\ \mathbf{l} \times (\mathbf{n} \times \mathbf{m})\ ]_2 \\
+i\ [\ m_0(\mathbf{n} \times \mathbf{l})_2 + l_0(\mathbf{n} \times \mathbf{m})_2 + n_0(\mathbf{l} \times \mathbf{m})_2\ ]\ .
\end{aligned} \tag{28}
$$

Thus, parameter $(k_a)^{-1}$ is defined as follows:

$$
\begin{aligned}
(k_0)^{-1} = k_0\ (mm) + m_0\ (ln) + l_0\ (nm) - n_0(lm) + i\ \mathbf{l}\ (\mathbf{m} \times \mathbf{n})\ , \\
(k_j)^{-1} = -k_j\ (mm) - m_j\ (ln) - l_j\ (nm) + n_j(lm) \\
+2\ [\ \mathbf{l} \times (\mathbf{n} \times \mathbf{m})\ ]_j + i\ [\ m_0(\mathbf{n} \times \mathbf{l})_j + l_0(\mathbf{n} \times \mathbf{m})_j + n_0(\mathbf{l} \times \mathbf{m})_j\ ]\ .
\end{aligned} \tag{29}
$$

One may expect to obtain similar formulas for quantities $m, l, n$.
Now let us calculate

$$
(n_0)^{-1} = \frac{1}{2\mid G\mid}\ (A_{42} + A_{31})\ , \qquad (n_3)^{-1} = \frac{1}{2\mid G\mid}\ (A_{42} - A_{31})\ . \tag{30}
$$

Cofactor $A_{42}$ is

$$
\begin{aligned}
A_{42} = \left| \begin{array}{ccc} +(k_0+k_3) & +(n_0-n_3) & -(n_1-in_2) \\ +(k_1+ik_2) & -(n_1+in_2) & +(n_0+n_3) \\ -(l_0+l_3) & +(m_0-m_3) & -(m_1-im_2) \end{array} \right| = \\
= -(l_0+l_3)\ (nn) + (k_0+k_3)(n_1+in_2)(m_1-im_2) \\
-(m_0-m_3)(k_1+ik_2)(n_1-in_2) \\
+(n_0-n_3)(k_1+k_2)(m_1-im_2) \\
-(k_0+k_3)(m_0-m_3)(n_0+n_3)\ .
\end{aligned} \tag{31}
$$

Cofactor $A_{31}$ is

$$
\begin{aligned}
A_{31} = \left| \begin{array}{ccc} +(k_1-ik_2) & +(n_0-n_3) & -(n_1-in_2) \\ +(k_0-k_3) & -(n_1+in_2) & +(n_0+n_3) \\ -(l_0-l_3) & -(m_1+im_2) & +(m_0+m_3) \end{array} \right| = \\
= -(l_0-l_3)\ (nn) + (k_0-k_3)(n_1-in_2)(m_1+im_2) \\
-(m_0+m_3)(k_1-ik_2)(n_1+in_2) \\
+(n_0+n_3)(k_1-k_2)(m_1+im_2) \\
-(k_0-k_3)(m_0+m_3)(n_0-n_3)\ .
\end{aligned} \tag{32}
$$

With the use of relations:

$$\frac{1}{2}\,[\,(k_0+k_3)(n_1+in_2)(m_1-im_2)+(k_0-k_3)(n_1-in_2)(m_1+im_2)\,]$$
$$=k_0n_1m_1+k_0n_2m_2-ik_3n_1m_2+ik_3n_2m_1\ ,$$
$$\frac{1}{2}\,[\,(k_0+k_3)(n_1+in_2)(m_1-im_2)-(k_0-k_3)(n_1-in_2)(m_1+im_2)\,]$$
$$=-ik_0n_1m_2+ik_0n_2m_1+k_3n_1m_1+k_3n_2m_2\ ,$$

$$-\frac{1}{2}\,[\,(m_0-m_3)(k_1+ik_2)(n_1-in_2)+(m_0+m_3)(k_1-ik_2)(n_1+in_2)\,]$$
$$=-m_0k_1n_1-m_0k_2n_2-im_3k_1n_2+im_3k_2n_1\ ,$$
$$-\frac{1}{2}\,[\,(m_0-m_3)(k_1+ik_2)(n_1-in_2)-(m_0+m_3)(k_1-ik_2)(n_1+in_2)\,]$$
$$=+im_0k_1n_2-im_0k_2n_1+m_3k_1n_1+m_3k_2n_2\ ,$$

$$\frac{1}{2}\,[\,(n_0-n_3)(k_1+k_2)(m_1-im_2)+(n_0+n_3)(k_1-k_2)(m_1+im_2)\,]$$
$$=n_0k_1m_1+n_0k_2m_2+in_3k_1m_2-in_3k_2m_1\ ,$$
$$\frac{1}{2}\,[\,(n_0-n_3)(k_1+k_2)(m_1-im_2)-(n_0+n_3)(k_1-k_2)(m_1+im_2)\,]$$
$$=-in_0k_1m_2+in_0k_2m_1-n_3k_1m_1-n_3k_2m_2\ ,$$

$$-\frac{1}{2}\,[\,(k_0+k_3)(m_0-m_3)(n_0+n_3)+(k_0-k_3)(m_0+m_3)(n_0-n_3)\,]$$
$$=-k_0m_0n_0+k_0m_3n_3-k_3m_0n_3+k_3m_3n_0\ ,$$
$$-\frac{1}{2}\,[\,(k_0+k_3)(m_0-m_3)(n_0+n_3)-(k_0-k_3)(m_0+m_3)(n_0-n_3)\,]$$
$$=-k_0m_0n_3+k_0m_3n_0-k_3m_0n_0+k_3m_3n_3\ ,$$

we arrive at (factor $\mid G\mid^{-1}$ is omitted)

$$\begin{aligned}(n_0)^{-1} &= -l_0\ (nn)\\ &+k_0n_1m_1+k_0n_2m_2-ik_3n_1m_2+ik_3n_2m_1\\ &-m_0k_1n_1-m_0k_2n_2-im_3k_1n_2+im_3k_2n_1\\ &+n_0k_1m_1+n_0k_2m_2+in_3k_1m_2-in_3k_2m_1\\ &-k_0m_0n_0+k_0m_3n_3-k_3m_0n_3+k_3m_3n_0\ ,\end{aligned}$$

$$\begin{aligned}(n_3)^{-1} &= -l_3\ (nn)\\ &-ik_0n_1m_2+ik_0n_2m_1+k_3n_1m_1+k_3n_2m_2\\ &+im_0k_1n_2-im_0k_2n_1+m_3k_1n_1+m_3k_2n_2\\ &-in_0k_1m_2+in_0k_2m_1-n_3k_1m_1-n_3k_2m_2\\ &-k_0m_0n_3+k_0m_3n_0-k_3m_0n_0+k_3m_3n_3\ .\end{aligned}$$

From this it follows:

$$(n_0)^{-1} = -k_0\ (nm) + m_0\ (kn) - l_0\ (nn) - n_0\ (km)\ +\ i\ \mathbf{k}\ (\mathbf{m} \times \mathbf{n})\ ,$$

$$(n_3)^{-1} = -k_3\ (nm) + m_3\ (kn) - l_3\ (nn) - n_3\ (km)$$
$$+\ 2\ [\ \mathbf{k} \times (\mathbf{m} \times \mathbf{n})\ ]_3\ +\ ik_0\ (\mathbf{m} \times \mathbf{n})_3 + im_0\ (\mathbf{k} \times \mathbf{n})_3 + in_0\ (\mathbf{m} \times \mathbf{k})_3\ . \qquad (33)$$

Now, let us calculate

$$-\ (n_1)^{-1} = \frac{A_{41} + A_{32}}{2\mid G \mid}\ , \qquad i(n_2)^{-1} = \frac{A_{41} - A_{32}}{2\mid G \mid}\ . \qquad (34)$$

Cofactor $A_{41}$ is

$$A_{41} = (-1) \begin{vmatrix} +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \end{vmatrix} =$$
$$= (-1)\{\ -(l_1 - il_2)\ (nn) + (k_1 - ik_2)(n_1 + in_2)(m_1 - im_2)$$
$$-(k_0 - k_3)(m_0 - m_3)(n_1 - in_2) +$$
$$+(k_0 - k_3)(n_0 - n_3)(m_1 - im_2)$$
$$-(m_0 - m_3)(n_0 + n_3)(k_1 - ik_2)\ \}\ , \qquad (35)$$

Cofactor $A_{32}$ is

$$A_{32} = (-1) \begin{vmatrix} +(k_0 + k_3) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_1 + il_2) & -(m_1 + im_2) & +(m_0 + m_3) \end{vmatrix} =$$
$$= (-1)\ \{\ -(l_1 + il_2)\ (nn) + (k_1 + ik_2)(n_1 - in_2)(m_1 + im_2)$$
$$-(k_0 + k_3)(m_0 + m_3)(n_1 + in_2)$$
$$+(k_0 + k_3)(n_0 + n_3)(m_1 + im_2)$$
$$-(m_0 + m_3)(n_0 - n_3)(k_1 + ik_2)\ \}\ . \qquad (36)$$

Using the identities:

$$-\ \frac{1}{2}\ [\ (k_1 - ik_2)(n_1 + in_2)(m_1 - im_2) + (k_1 + ik_2)(n_1 - in_2)(m_1 + im_2)\ ]$$
$$= -k_1 n_1 m_1 - k_1 n_2 m_2 + k_2 n_1 m_2 - k_2 n_2 m_1\ ,$$
$$-\frac{1}{2}\ [\ (k_1 - ik_2)(n_1 + in_2)(m_1 - im_2) - (k_1 + ik_2)(n_1 - in_2)(m_1 + im_2)\ ]$$
$$= +ik_1 n_1 m_2 - ik_1 n_2 m_1 + ik_2 n_1 m_1 + ik_2 n_2 m_2\ ,$$

$$\frac{1}{2}\ [\ (k_0 - k_3)(m_0 - m_3)(n_1 - in_2) + (k_0 + k_3)(m_0 + m_3)(n_1 + in_2)\ ]$$
$$= k_0 m_0 n_1 + ik_0 m_3 n_2 + ik_3 m_0 n_2 + k_3 m_3 n_1\ ,$$
$$\frac{1}{2}\ [\ (k_0 - k_3)(m_0 - m_3)(n_1 - in_2) - (k_0 + k_3)(m_0 + m_3)(n_1 + in_2)\ ]$$
$$= -ik_0 m_0 n_2 - k_0 m_3 n_1 - k_3 m_0 n_1 - ik_3 m_3 n_2\ ,$$

$$-\frac{1}{2}\,[\,(k_0-k_3)(n_0-n_3)(m_1-im_2)+(k_0+k_3)(n_0+n_3)(m_1+im_2)\,]$$
$$= -k_0n_0m_1-k_3n_3m_1-ik_0n_3m_2-ik_3n_0m_2\;,$$
$$-\frac{1}{2}\,[\,(k_0-k_3)(n_0-n_3)(m_1-im_2)-(k_0+k_3)(n_0+n_3)(m_1+im_2)\,]$$
$$= +k_0n_3m_1+k_3n_0m_1+ik_0n_0m_2+ik_3n_3m_2\;,$$

$$\frac{1}{2}\,[\,(m_0-m_3)(n_0+n_3)(k_1-ik_2)+(m_0+m_3)(n_0-n_3)(k_1+ik_2)\,]$$
$$= m_0n_0k_1-m_3n_3k_1-im_0n_3k_2+im_3n_0k_2\;,$$
$$\frac{1}{2}\,[\,(m_0-m_3)(n_0+n_3)(k_1-ik_2)-(m_0+m_3)(n_0-n_3)(k_1+ik_2)\,]$$
$$= m_0n_3k_1-m_3n_0k_1-im_0n_0k_2+im_3n_3k_2\;,$$

we get expressions for $(n_1)^{-1}$ and $(n_2)^{-1}$:

$$\begin{aligned}
-\,(n_1)^{-1} &= l_1\;(nn) \\
&-k_1n_1m_1-k_1n_2m_2+k_2n_1m_2-k_2n_2m_1 \\
&+k_0m_0n_1+ik_0m_3n_2+ik_3m_0n_2+k_3m_3n_1 \\
&-k_0n_0m_1-k_3n_3m_1-ik_0n_3m_2-ik_3n_0m_2 \\
&+m_0n_0k_1-m_3n_3k_1-im_0n_3k_2+im_3n_0k_2\;,
\end{aligned}$$

$$\begin{aligned}
i(n_2)^{-1} &= -il_2\;(nn) \\
&+ik_1n_1m_2-ik_1n_2m_1+ik_2n_1m_1+ik_2n_2m_2 \\
&-ik_0m_0n_2-k_0m_3n_1-k_3m_0n_1-ik_3m_3n_2 \\
&+k_0n_3m_1+k_3n_0m_1+ik_0n_0m_2+ik_3n_3m_2 \\
&+m_0n_3k_1-m_3n_0k_1-im_0n_0k_2+im_3n_3k_2\;.
\end{aligned}$$

From where, after identical transformations we arrive at

$$\begin{aligned}
&(n_1)^{-1} = -k_1\;(nm)+m_1\;(kn)-l_1\;(nn)-n_1\;(km) \\
&+2\,[\,\mathbf{k}\times(\mathbf{m}\times\mathbf{n})\,]_1\;+\;ik_0\;(\mathbf{m}\times\mathbf{n})_1+im_0\;(\mathbf{k}\times\mathbf{n})_1+in_0\;(\mathbf{m}\times\mathbf{k})_2\;, \\
&(n_2)^{-1} = -k_2\;(nm)+m_2\;(kn)-l_2\;(nn)-n_2\;(km) \\
&+2\,[\,\mathbf{k}\times(\mathbf{m}\times\mathbf{n})\,]_2\;+\;ik_0\;(\mathbf{m}\times\mathbf{n})_2+im_0\;(\mathbf{k}\times\mathbf{n})_2+in_0\;(\mathbf{m}\times\mathbf{k})_2\;.
\end{aligned} \tag{37}$$

Thus, parameter $(n_a)^{-1}$ is defined by

$$\begin{aligned}
&(n_0)^{-1} = -k_0\;(nm)+m_0\;(kn)-l_0\;(nn)-n_0\;(km)\;+\;i\,\mathbf{k}\,(\mathbf{m}\times\mathbf{n})\;, \\
&(n_j)^{-1} = -k_j\;(nm)+m_j\;(kn)-l_j\;(nn)-n_j\;(km) \\
&+2\,[\,\mathbf{k}\times(\mathbf{m}\times\mathbf{n})\,]_j\;+\;ik_0\;(\mathbf{m}\times\mathbf{n})_j+im_0\;(\mathbf{k}\times\mathbf{n})_j+in_0\;(\mathbf{m}\times\mathbf{k})_j\;.
\end{aligned} \tag{38}$$

Let us calculate

$$-\,(l_0)^{-1} = \frac{A_{13}+A_{24}}{2\mid G\mid}\;, \qquad -(l_3)^{-1} = \frac{A_{13}-A_{24}}{2\mid G\mid}\;. \tag{39}$$

Cofactor $A_{13}$ is

$$A_{13} = \left| \begin{array}{ccc} +(k_1+ik_2) & +(k_0-k_3) & +(n_0+n_3) \\ -(l_0+l_3) & -(l_1-il_2) & -(m_1-im_2) \\ -(l_1+il_2) & -(l_0-l_3) & +(m_0+m_3) \end{array} \right| =$$

$$\begin{aligned} &= (n_0+n_3)\,(ll) - (m_0+m_3)(k_1+ik_2)(l_1-il_2) \\ &\quad +(k_0-k_3)(m_1-im_2)(l_1+il_2) \\ &\quad +(l_0+l_3)((k_0-k_3)(m_0+m_3) \\ &\quad -(l_0-l_3)(k_1+ik_2)(m_1-im_2)\,. \end{aligned} \tag{40}$$

Cofactor $A_{24}$ is

$$A_{24} = \left| \begin{array}{ccc} +(k_0+k_3) & +(k_1-ik_2) & +(n_0-n_3) \\ -(l_0+l_3) & -(l_1-il_2) & +(m_0-m_3) \\ -(l_1+il_2) & -(l_0-l_3) & -(m_1+im_2) \end{array} \right| =$$

$$\begin{aligned} &= (n_0-n_3)\,(ll) - (m_0-m_3)(k_1-ik_2)(l_1+il_2) \\ &\quad +(k_0+k_3)(m_1+im_2)(l_1-il_2)+ \\ &\quad +(l_0-l_3)((k_0+k_3)(m_0-m_3) \\ &\quad -(l_0+l_3)(k_1-ik_2)(m_1+im_2)\,. \end{aligned} \tag{41}$$

Using the relations:

$$\begin{aligned} &-\frac{1}{2}[\,(m_0+m_3)(k_1+ik_2)(l_1-il_2)+(m_0-m_3)(k_1-ik_2)(l_1+il_2)\,] \\ &\qquad = -m_0k_1l_1 - m_0k_2l_2 + im_3k_1l_2 - im_3k_2l_1\,, \\ &-\frac{1}{2}[\,(m_0+m_3)(k_1+ik_2)(l_1-il_2)-(m_0-m_3)(k_1-ik_2)(l_1+il_2)\,] \\ &\qquad = +im_0k_1l_2 - im_0k_2l_1 + m_3k_1l_1 - m_3k_2l_2\,, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\,[\,(k_0-k_3)(m_1-im_2)(l_1+il_2)+(k_0+k_3)(m_1+im_2)(l_1-il_2)\,] \\ &\qquad = k_0m_1l_1 + k_0m_2l_2 - ik_3m_1l_2 + ik_3m_2l_1\,, \\ &\frac{1}{2}\,[\,(k_0-k_3)(m_1-im_2)(l_1+il_2)-(k_0+k_3)(m_1+im_2)(l_1-il_2)\,] \\ &\qquad = +ik_0m_1l_2 - ik_0m_2l_1 - k_3m_1l_1 - k_3m_2l_2\,, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}[\,(l_0+l_3)((k_0-k_3)(m_0+m_3)+(l_0-l_3)((k_0+k_3)(m_0-m_3)\,] \\ &\qquad = l_0k_0m_0 - l_0k_3m_3 + l_3k_0m_3 - l_3k_3m_0\,, \\ &\frac{1}{2}[\,(l_0+l_3)((k_0-k_3)(m_0+m_3)-(l_0-l_3)((k_0+k_3)(m_0-m_3)\,] \\ &\qquad = l_0k_0m_3 - l_0k_3m_0 + l_3k_0m_0 - l_3k_3m_3\,, \end{aligned}$$

$$\begin{aligned} &-\frac{1}{2}[\,(l_0-l_3)(k_1+ik_2)(m_1-im_2)+(l_0+l_3)(k_1-ik_2)(m_1+im_2)\,] \\ &\qquad = -l_0k_1m_1 - l_0k_2m_2 - il_3k_1m_2 + il_3k_2m_1\,, \\ &-\frac{1}{2}[\,(l_0-l_3)(k_1+ik_2)(m_1-im_2)-(l_0+l_3)(k_1-ik_2)(m_1+im_2)\,] \\ &\qquad = +il_0k_1m_2 - il_0k_2m_1 + l_3k_1m_1 + l_3k_2m_2\,, \end{aligned}$$

we get

$$
\begin{aligned}
-\,(l_0)^{-1} &= +n_0\ (ll) \\
&-m_0k_1l_1 - m_0k_2l_2 + im_3k_1l_2 - im_3k_2l_1 \\
&+k_0m_1l_1 + k_0m_2l_2 - ik_3m_1l_2 + ik_3m_2l_1 \\
&+l_0k_0m_0 - l_0k_3m_3 + l_3k_0m_3 - l_3k_3m_0 \\
&-l_0k_1m_1 - l_0k_2m_2 - il_3k_1m_2 + il_3k_2m_1\ ,
\end{aligned}
$$

$$
\begin{aligned}
-\,(l_3)^{-1} &= +n_3\ (ll) \\
&+im_0k_1l_2 - im_0k_2l_1 + m_3k_1l_1 - m_3k_2l_2 \\
&+ik_0m_1l_2 - ik_0m_2l_1 - k_3m_1l_1 - k_3m_2l_2 \\
&+l_0k_0m_3 - l_0k_3m_0 + l_3k_0m_0 - l_3k_3m_3 \\
&+il_0k_1m_2 - il_0k_2m_1 + l_3k_1m_1 + l_3k_2m_2\ .
\end{aligned}
$$

From where we arrive at

$$
(l_0)^{-1} = +k_0\ (ml) - m_0\ (kl) - l_0\ (km) - n_0\ (ll)\ +\ i\ \mathbf{m}\ (\mathbf{l}\times\mathbf{k})\ ,
$$

$$
\begin{aligned}
(l_3)^{-1} &= +k_3\ (ml) - m_3\ (kl) - l_3\ (km) - n_3\ (ll) \\
&+2\ [\mathbf{m}\times(\mathbf{k}\times\mathbf{l})\ ]_3\ +\ i\ m_0\ (\mathbf{l}\times\mathbf{k})_3 + i\ k_0\ (\mathbf{l}\times\mathbf{m})_3 + i\ l_0\ (\mathbf{m}\times\mathbf{k})_3\ .
\end{aligned} \tag{42}
$$

Now let us calculate

$$
-\,(l_1)^{-1} = \frac{A_{23}+A_{14}}{2\mid G\mid}\ , \qquad i(l_2)^{-1} = \frac{A_{23}-A_{14}}{2\mid G\mid}\ . \tag{43}
$$

Cofactor $A_{23}$ is

$$
\begin{aligned}
A_{23} = (-1)&\begin{vmatrix} +(k_0+k_3) & +(k_1-ik_2) & -(n_1-in_2) \\ -(l_0+l_3) & -(l_1-il_2) & -(m_1-im_2) \\ -(l_1+il_2) & -(l_0-l_3) & +(m_0+m_3) \end{vmatrix} = \\
&(-1)\ [\ -(n_1-in_2)\ (ll) - (k_0+k_3)(m_0+m_3)(l_1-il_2) \\
&+(l_0+l_3)(m_0+m_3)(k_1-ik_2) \\
&-(l_0-l_3)(k_0+k_3)(m_1-im_2) \\
&+(k_1-ik_2)(m_1-im_2)(l_1+il_2)\ ]\ ,
\end{aligned} \tag{44}
$$

Cofactor $A_{14}$ is

$$
\begin{aligned}
A_{14} = (-1)&\begin{vmatrix} +(k_1+ik_2) & +(k_0-k_3) & -(n_1+in_2) \\ -(l_0+l_3) & -(l_1-il_2) & +(m_0-m_3) \\ -(l_1+il_2) & -(l_0-l_3) & -(m_1+im_2) \end{vmatrix} = \\
&= (-1)\ [\ -(n_1+in_2)\ (ll) \\
&-(k_0-k_3)(m_0-m_3)(l_1+il_2) \\
&+(l_0-l_3)(m_0-m_3)(k_1+ik_2) \\
&-(l_0+l_3)(k_0-k_3)(m_1+im_2) \\
&+(k_1+ik_2)(m_1+im_2)(l_1-il_2)\ ]\ .
\end{aligned} \tag{45}
$$

With the use of identities:

$$\frac{1}{2}\,[\,(k_0+k_3)(m_0+m_3)(l_1-il_2)+(k_0-k_3)(m_0-m_3)(l_1+il_2)\,]$$
$$=l_1k_0m_0+l_1k_3m_3-il_2k_0m_3-il_2k_3m_0\ ,$$
$$\frac{1}{2}\,[\,(k_0+k_3)(m_0+m_3)(l_1-il_2)-(k_0-k_3)(m_0-m_3)(l_1+il_2)\,]$$
$$=l_1k_0m_3+l_1k_3m_0-il_2k_0m_0-il_2k_3m_3\ ,$$

$$-\frac{1}{2}\,[\,(l_0+l_3)(m_0+m_3)(k_1-ik_2)+(l_0-l_3)(m_0-m_3)(k_1+ik_2)\,]$$
$$=-k_1l_0m_0-k_1l_3m_3+ik_2l_0m_3+ik_2l_3m_0\ ,$$
$$-\frac{1}{2}\,[\,(l_0+l_3)(m_0+m_3)(k_1-ik_2)-(l_0-l_3)(m_0-m_3)(k_1+ik_2)\,]$$
$$=-k_1l_0m_3-k_1l_3m_0+ik_2l_0m_0+ik_2l_3m_3\ ,$$

$$\frac{1}{2}\,[\,(l_0-l_3)(k_0+k_3)(m_1-im_2)+(l_0+l_3)(k_0-k_3)(m_1+im_2)\,]$$
$$=m_1l_0k_0-m_1l_3k_3-im_2l_0k_3+im_2l_3k_0\ ,$$
$$\frac{1}{2}\,[\,(l_0-l_3)(k_0+k_3)(m_1-im_2)-(l_0+l_3)(k_0-k_3)(m_1+im_2)\,]$$
$$=m_1l_0k_3-m_1l_3k_0-im_2l_0k_0+im_2l_3k_3\ ,$$

$$-\frac{1}{2}\,[\,(k_1-ik_2)(m_1-im_2)(l_1+il_2)+(k_1+ik_2)(m_1+im_2)(l_1-il_2)\,]$$
$$=-k_1m_1l_1-k_1m_2l_2-k_2m_1l_2+k_2m_2l_1\ ,$$
$$-\frac{1}{2}\,[\,(k_1-ik_2)(m_1-im_2)(l_1+il_2)-(k_1+ik_2)(m_1+im_2)(l_1-il_2)\,]$$
$$=-ik_1m_1l_2+ik_1m_2l_1+ik_2m_1l_1+ik_2m_2l_2\ ,$$

we get expressions for $(l_1)^{-1}$ and $(l_2)^{-1}$:

$$\begin{aligned}
-\,(l_1)^{-1} &= +n_1\,(ll)\\
+l_1k_0m_0+l_1k_3m_3-il_2k_0m_3-il_2k_3m_0 &-\\
-k_1l_0m_0-k_1l_3m_3+ik_2l_0m_3+ik_2l_3m_0 &\\
+m_1l_0k_0-m_1l_3k_3-im_2l_0k_3+im_2l_3k_0 &\\
-k_1m_1l_1-k_1m_2l_2-k_2m_1l_2+k_2m_2l_1 &\ ,
\end{aligned}$$

$$\begin{aligned}
i\,(l_2)^{-1} &= -i\,n_2\,(ll)\\
+l_1k_0m_3+l_1k_3m_0-il_2k_0m_0-il_2k_3m_3 &\\
-k_1l_0m_3-k_1l_3m_0+ik_2l_0m_0+ik_2l_3m_3 &\\
+m_1l_0k_3-m_1l_3k_0-im_2l_0k_0+im_2l_3k_3 &\\
-ik_1m_1l_2+ik_1m_2l_1+ik_2m_1l_1+ik_2m_2l_2 &\ .
\end{aligned}$$

From where it follows

$$(l_1)^{-1} = +k_1\ (ml) - m_1\ (kl) - l_1\ (km) - n_1\ (ll)$$
$$+2\ [\mathbf{m} \times (\mathbf{k} \times \mathbf{l})\ ]_1\ +\ i\ m_0\ (\mathbf{l} \times \mathbf{k})_1 + i\ k_0\ (\mathbf{l} \times \mathbf{m})_1 + i\ l_0\ (\mathbf{m} \times \mathbf{k})_1\ ,$$
$$(l_2)^{-1} = +k_2\ (ml) - m_2\ (kl) - l_2\ (km) - n_3\ (ll)$$
$$+2\ [\mathbf{m} \times (\mathbf{k} \times \mathbf{l})\ ]_2\ +\ i\ m_0\ (\mathbf{l} \times \mathbf{k})_2 + i\ k_0\ (\mathbf{l} \times \mathbf{m})_2 + i\ l_0\ (\mathbf{m} \times \mathbf{k})_2\ . \qquad (46)$$

Thus, parameter $(l)^{-1}$ is defined by

$$(l_0)^{-1} = +k_0\ (ml) - m_0\ (kl) - l_0\ (km) - n_0\ (ll)\ +\ i\ \mathbf{m}\ (\mathbf{l} \times \mathbf{k})\ ,$$
$$(l_j)^{-1} = +k_j\ (ml) - m_j\ (kl) - l_3\ (km) - n_j\ (ll)$$
$$+2\ [\mathbf{m} \times (\mathbf{k} \times \mathbf{l})\ ]_j\ +\ i\ m_0\ (\mathbf{l} \times \mathbf{k})_j + i\ k_0\ (\mathbf{l} \times \mathbf{m})_j + i\ l_0\ (\mathbf{m} \times \mathbf{k})_j\ . \qquad (47)$$

It remains to calculate parameter $(m)^{-1}$. For $(m_0)^{-1}$ and $(m_3)^{-1}$ we have

$$(m_0)^{-1} = \frac{A_{44} + A_{33}}{2\mid G \mid}\ , \qquad (m_3)^{-1} = \frac{A_{44} - A_{33}}{2\mid G \mid}\ . \qquad (48)$$

Cofactor $A_{44}$ is

$$A_{44} = \left| \begin{array}{ccc} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) \end{array} \right| =$$
$$= (m_0 - m_3)\ (kk) - (n_0 - n_3)(k_1 + ik_2)(l_1 - il_2)$$
$$+(l_0 + l_3)(k_1 - ik_2)(n_1 + in_2)$$
$$+(l_0 + l_3)((k_0 - k_3)(n_0 - n_3)$$
$$-(k_0 + k_3)(l_1 - il_2)(n_1 + in_2)\ , \qquad (49)$$

Cofactor $A_{33}$ is

$$A_{33} = \left| \begin{array}{ccc} +(k_0 + k_3) & +(k_1 - ik_2) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & +(n_0 + n_3) \\ -(l_1 + il_2) & -(l_0 - l_3) & +(m_0 + m_3) \end{array} \right| =$$
$$= (m_0 + m_3)\ (kk) - (n_0 + n_3)(k_1 - ik_2)(l_1 + il_2)$$
$$+(l_0 - l_3)(k_1 + ik_2)(n_1 - in_2)$$
$$+(l_0 - l_3)((k_0 + k_3)(n_0 + n_3)$$
$$-(k_0 - k_3)(l_1 + il_2)(n_1 - in_2)\ . \qquad (50)$$

With the help of identities

$$-\frac{1}{2}\ [\ (n_0 - n_3)(k_1 + ik_2)(l_1 - il_2) + (n_0 + n_3)(k_1 - ik_2)(l_1 + il_2)\ ]$$
$$= -n_0 k_1 l_1 - n_0 k_2 l_2 - in_3 k_1 l_2 + in_3 k_2 l_1\ ,$$
$$-\frac{1}{2}\ [\ (n_0 - n_3)(k_1 + ik_2)(l_1 - il_2) - (n_0 + n_3)(k_1 - ik_2)(l_1 + il_2)\ ]$$
$$= +in_0 k_1 l_2 - in_0 k_2 l_1 + n_3 k_1 l_1 + n_3 k_2 l_2\ ,$$

$$\frac{1}{2}\,[\,(l_0+l_3)(k_1-ik_2)(n_1+in_2)+(l_0-l_3)(k_1+ik_2)(n_1-in_2)\,]$$
$$= l_0k_1n_1+l_0k_2n_2+il_3k_1n_2-il_3k_2n_1\ ,$$
$$\frac{1}{2}\,[\,(l_0+l_3)(k_1-ik_2)(n_1+in_2)-(l_0-l_3)(k_1+ik_2)(n_1-in_2)\,]$$
$$= il_0k_1n_2-il_0k_2n_1+l_3k_1n_1+l_3k_2n_2\ ,$$

$$\frac{1}{2}\,[\,(l_0+l_3)(k_0-k_3)(n_0-n_3)+(l_0-l_3)(k_0+k_3)(n_0+n_3)\,]$$
$$= l_0k_0n_0+l_0k_3n_3-l_3k_0n_3-l_3k_3n_0\ ,$$
$$\frac{1}{2}\,[\,(l_0+l_3)(k_0-k_3)(n_0-n_3)-(l_0-l_3)(k_0+k_3)(n_0+n_3)\,]$$
$$= -l_0k_0n_3-l_0k_3n_0+l_3k_0n_0+l_3k_3n_3\ ,$$

$$-\,\frac{1}{2}\,[\,(k_0+k_3)(l_1-il_2)(n_1+in_2)+(k_0-k_3)(l_1+il_2)(n_1-in_2)\,]$$
$$= -k_0l_1n_1-k_0l_2n_2-ik_3l_1n_2+ik_3l_2n_1\ ,$$
$$-\frac{1}{2}\,[\,(k_0+k_3)(l_1-il_2)(n_1+in_2)-(k_0-k_3)(l_1+il_2)(n_1-in_2)\,]$$
$$= -ik_0l_1n_2+ik_0l_2n_1-k_3l_1n_1+k_3l_2n_2\ .$$

we get expressions for $(m_0)^{-1}$ and $(m_3)^{-1}$:

$$\begin{aligned}(m_0)^{-1} &= +m_0\ (kk)\\ -n_0k_1l_1-n_0k_2l_2-in_3k_1l_2+in_3k_2l_1\\ +l_0k_1n_1+l_0k_2n_2+il_3k_1n_2-il_3k_2n_1\\ +l_0k_0n_0+l_0k_3n_3-l_3k_0n_3-l_3k_3n_0\\ -k_0l_1n_1-k_0l_2n_2-ik_3l_1n_2+ik_3l_2n_1\ ,\end{aligned}$$

$$\begin{aligned}(m_3)^{-1} &= -m_3\ (kk)\\ +in_0k_1l_2-in_0k_2l_1+n_3k_1l_1+n_3k_2l_2\\ +il_0k_1n_2-il_0k_2n_1+l_3k_1n_1+l_3k_2n_2\\ -l_0k_0n_3-l_0k_3n_0+l_3k_0n_0+l_3k_3n_3\\ -ik_0l_1n_2+ik_0l_2n_1-k_3l_1n_1+k_3l_2n_2\ .\end{aligned}$$

From where we arrive at

$$(m_0)^{-1} = k_0\ (ln)+m_0\ (kk)-l_0(kn)+n_0\ lk)\ +\ i\,\mathbf{n}\,(\mathbf{l}\times\mathbf{k})\ ,$$
$$(m_3)^{-1} = -k_3\ (ln)-m_3\ (kk)+l_3\ (kn)-n_3\ (kl)$$
$$+2\,[\,\mathbf{n}\times(\mathbf{l}\times\mathbf{k})\,]_3\ +\ i\ n_0\ (\mathbf{k}\times\mathbf{l})_3+i\ l_0\ (\mathbf{k}\times\mathbf{n})_3+i\ k_0\ (\mathbf{n}\times\mathbf{l})_3\ . \tag{51}$$

Now let us calculate $(m_1)^{-1}$ and $(m_2)^{-1}$:

$$-\,(m_1)^{-1} = \frac{A_{43}+A_{34}}{2\mid G\mid}\ ,\qquad i(m_2)^{-1} = \frac{A_{43}-A_{34}}{2\mid G\mid}\ . \tag{52}$$

Cofactor $A_{43}$ is

$$A_{43} = (-1) \begin{vmatrix} +(k_0+k_3) & +(k_1-ik_2) & -(n_1-in_2) \\ +(k_1+ik_2) & +(k_0-k_3) & +(n_0+n_3) \\ -(l_0+l_3) & -(l_1-il_2) & -(m_1-im_2) \end{vmatrix} =$$

$$\begin{aligned} = (-1)\,[\, -(m_1-im_2)\,(kk) &+ (k_1+ik_2)(n_1-in_2)(l_1-il_2) \\ &-(k_1-ik_2)(n_0+n_3)(l_0+l_3) \\ &-(n_1-in_2)(l_0+l_3)(k_0-k_3) \\ &+(l_1-il_2)(k_0+k_3)(n_0+n_3)\,]\,. \end{aligned} \qquad (53)$$

Cofactor $A_{43}$ is

$$A_{34} = \begin{vmatrix} +(k_0+k_3) & +(k_1-ik_2) & +(n_0-n_3) \\ +(k_1+ik_2) & +(k_0-k_3) & -(n_1+in_2) \\ -(l_1+il_2) & -(l_0-l_3) & -(m_1+im_2) \end{vmatrix} =$$

$$\begin{aligned} = (-1)\,[\, -(m_1+im_2)\,(kk) &+ (k_1-ik_2)(n_1+in_2)(l_1+il_2) \\ &-(k_1+ik_2)(n_0-n_3)(l_0-l_3) \\ &-(n_1+in_2)(l_0-l_3)(k_0+k_3) \\ &+(l_1+il_2)(k_0-k_3)(n_0-n_3)\,]\,. \end{aligned} \qquad (54)$$

With the use of relations:

$$-\frac{1}{2}\,[\,(k_1+ik_2)(n_1-in_2)(l_1-il_2) + (k_1-ik_2)(n_1+in_2)(l_1+il_2)\,]$$
$$= -k_1n_1l_1 + k_1n_2l_2 - k_2n_1l_2 - k_2n_2l_1\,,$$

$$-\frac{1}{2}\,[\,(k_1+ik_2)(n_1-in_2)(l_1-il_2) - (k_1-ik_2)(n_1+in_2)(l_1+il_2)\,]$$
$$= +ik_1n_1l_2 + ik_1n_2l_1 - ik_2n_1l_1 + ik_2n_2l_2\,,$$

$$\frac{1}{2}\,[\,(k_1-ik_2)(n_0+n_3)(l_0+l_3) + (k_1+ik_2)(n_0-n_3)(l_0-l_3)\,]$$
$$= k_1n_0l_0 + k_1n_3l_3 - ik_2n_0l_3 - ik_2n_3l_0\,,$$

$$\frac{1}{2}\,[\,(k_1-ik_2)(n_0+n_3)(l_0+l_3) - (k_1+ik_2)(n_0-n_3)(l_0-l_3)\,]$$
$$= k_1n_0l_3 + k_1n_3l_0 - ik_2n_0l_0 - ik_2n_3l_3\,,$$

$$\frac{1}{2}\,[\,(n_1-in_2)(l_0+l_3)(k_0-k_3) + (n_1+in_2)(l_0-l_3)(k_0+k_3)\,]$$
$$= n_1l_0k_0 - n_1l_3k_3 + in_2l_0k_3 - in_2l_3k_0\,,$$

$$\frac{1}{2}\,[\,(n_1-in_2)(l_0+l_3)(k_0-k_3) - (n_1+in_2)(l_0-l_3)(k_0+k_3)\,]$$
$$= -n_1l_0k_3 + n_1l_3k_0 - in_2l_0k_0 + in_2l_3k_3\,,$$

$$-\frac{1}{2}\,[\,(l_1-il_2)(k_0+k_3)(n_0+n_3) + (l_1+il_2)(k_0-k_3)(n_0-n_3)\,]$$
$$= -l_1k_0n_0 - l_1k_3n_3 + il_2k_0n_3 + il_2k_3n_0\,,$$

$$-\frac{1}{2}\,[\,(l_1-il_2)(k_0+k_3)(n_0+n_3) - (l_1+il_2)(k_0-k_3)(n_0-n_3)\,]$$
$$= -l_1k_0n_3 - l_1k_3n_0 + il_2k_0n_0 + il_2k_3n_3\,,$$

we get

$$
\begin{aligned}
-\,(m_1)^{-1} &= m_1\ (kk) \\
-k_1n_1l_1 + k_1n_2l_2 - k_2n_1l_2 - k_2n_2l_1 \\
+k_1n_0l_0 + k_1n_3l_3 - ik_2n_0l_3 - ik_2n_3l_0 \\
+n_1l_0k_0 - n_1l_3k_3 + in_2l_0k_3 - in_2l_3k_0 \\
-l_1k_0n_0 - l_1k_3n_3 + il_2k_0n_3 + il_2k_3n_0\ ,
\end{aligned}
$$

$$
\begin{aligned}
i(m_2)^{-1} &= -i\ m_2\ (kk) \\
+ik_1n_1l_2 + ik_1n_2l_1 - ik_2n_1l_1 + ik_2n_2l_2 \\
+k_1n_0l_3 + k_1n_3l_0 - ik_2n_0l_0 - ik_2n_3l_3 \\
-n_1l_0k_3 + n_1l_3k_0 - in_2l_0k_0 + in_2l_3k_3 \\
-l_1k_0n_3 - l_1k_3n_0 + il_2k_0n_0 + il_2k_3n_3\ .
\end{aligned}
$$

From where we arrive at

$$
\begin{aligned}
(m_1)^{-1} &= -k_1\ (ln) - m_1\ (kk) + l_1\ (kn) - n_1\ (kl) \\
+\ 2\ [\ \mathbf{n}\times(\mathbf{l}\times\mathbf{k})\ ]_1\ &+\ i\ n_0\ (\mathbf{k}\times\mathbf{l})_1 + i\ l_0\ (\mathbf{k}\times\mathbf{n})_1 + i\ k_0\ (\mathbf{n}\times\mathbf{l})_1\ , \\
(m_2)^{-1} &= -k_2\ (ln) - m_1\ (kk) + l_2\ (kn) - n_2\ (kl) \\
+\ 2\ [\ \mathbf{n}\times(\mathbf{l}\times\mathbf{k})\ ]_2\ &+\ i\ n_0\ (\mathbf{k}\times\mathbf{l})_2 + i\ l_0\ (\mathbf{k}\times\mathbf{n})_2 + i\ k_0\ (\mathbf{n}\times\mathbf{l})_2\ .
\end{aligned} \tag{55}
$$

Thus, the parameter $(m)^{-1}$ is defined by

$$
(m_0)^{-1} = k_0\ (ln) + m_0\ (kk) - l_0(kn) + n_0\ lk)\ +\ i\ \mathbf{n}\ (\mathbf{l}\times\mathbf{k})\ .
$$

$$
\begin{aligned}
(m_j)^{-1} &= -k_j\ (ln) - m_j\ (kk) + l_j\ (kn) - n_j\ (kl) \\
+2\ [\ \mathbf{n}\times(\mathbf{l}\times\mathbf{k})\ ]_j\ &+ i\ n_0\ (\mathbf{k}\times\mathbf{l})_j + i\ l_0\ (\mathbf{k}\times\mathbf{n})_j + i\ k_0\ (\mathbf{n}\times\mathbf{l})_j
\end{aligned} \tag{56}
$$

Dirac parameters for the inverse matrix $G^{-1}$ have been found; it remains to determine determinant of $G$.

## 4 Determinant $\mid G \mid$ in the Dirac parameters

Collecting all results on parameters of the inverse matrix $G^{-1}$ we have

$$
\begin{aligned}
k'_0 &= \mid G \mid^{-1}\ [\ k_0\ (mm) + m_0\ (ln) + l_0\ (nm) - n_0(lm) + i\ \mathbf{l}\ (\mathbf{m}\times\mathbf{n})\ ]\ , \\
\mathbf{k}' &= \mid G \mid^{-1}\ [\ -\mathbf{k}\ (mm) - \mathbf{m}\ (ln) - \mathbf{l}\ (nm)\ + \mathbf{n}(lm)\ + 2\ \mathbf{l}\times(\mathbf{n}\times\mathbf{m}) \\
&\qquad + i\ m_0(\mathbf{n}\times\mathbf{l})\ + i\ l_0(\mathbf{n}\times\mathbf{m}) + i\ n_0(\mathbf{l}\times\mathbf{m})\ ]\ , \\
m'_0 &= \mid G \mid^{-1}\ [\ k_0\ (ln) + m_0\ (kk) - l_0(kn) + n_0\ lk)\ + i\ \mathbf{n}\ (\mathbf{l}\times\mathbf{k})\ ]\ , \\
\mathbf{m}' &= \mid G \mid^{-1}\ [\ -\mathbf{k}\ (ln) - \mathbf{m}\ (kk) + \mathbf{l}\ (kn) - \mathbf{n}\ (kl)\ +\ 2\ \mathbf{n}\times(\mathbf{l}\times\mathbf{k}) \\
&\qquad +\ i\ n_0\ (\mathbf{k}\times\mathbf{l}) + i\ l_0\ (\mathbf{k}\times\mathbf{n}) + i\ k_0\ (\mathbf{n}\times\mathbf{l})\ ]\ ,
\end{aligned}
$$

$$
\begin{aligned}
l'_0 =& \mid G \mid^{-1} \ [ +k_0 \ (ml) - m_0 \ (kl) - l_0 \ (km) - n_0 \ (ll) \ + i \ \mathbf{m} \ (\mathbf{l} \times \mathbf{k}) \ ] \ , \\
\mathbf{l}' =& \mid G \mid^{-1} \ [ +\mathbf{k} \ (ml) - \mathbf{m} \ (kl) - \mathbf{l} \ (km) - \mathbf{n} \ (ll) \ + 2 \ \mathbf{m} \times (\mathbf{k} \times \mathbf{l}) \\
& + i \ m_0 \ (\mathbf{l} \times \mathbf{k}) + i \ k_0 \ (\mathbf{l} \times \mathbf{m}) + i \ l_0 \ (\mathbf{m} \times \mathbf{k}) \ ] \ , \\
n'_0 =& \mid G \mid^{-1} \ [ -k_0 \ (nm) + m_0 \ (kn) - l_0 \ (nn) - n_0 \ (km) \ + \ i \ \mathbf{k} \ (\mathbf{m} \times \mathbf{n}) \ ], \\
\mathbf{n}' =& \mid G \mid^{-1} \ [ -\mathbf{k} \ (nm) + \mathbf{m} \ (kn) - \mathbf{l} \ (nn) - \mathbf{n} \ (km) \ + \ 2 \ \mathbf{k} \times (\mathbf{m} \times \mathbf{n}) \\
& + ik_0 \ (\mathbf{m} \times \mathbf{n}) + im_0 \ (\mathbf{k} \times \mathbf{n}) + in_0 \ (\mathbf{m} \times \mathbf{k}) \ ] \ .
\end{aligned} \tag{57}
$$

Let us substitute these expressions for inverse parameters into determining relation $G^{-1}$, so we arrive at the equations:

$$1 = k''_0 = k'_0 \ k_0 + \mathbf{k}' \ \mathbf{k} - n'_0 \ l_0 + \mathbf{n}' \ \mathbf{l} \ , \tag{58}$$

$$0 = \mathbf{k}'' = k'_0 \ \mathbf{k} + \mathbf{k}' \ k_0 + i \ \mathbf{k}' \times \mathbf{k} - n'_0 \ \mathbf{l} + \mathbf{n}' \ l_0 + i \ \mathbf{n}' \times \mathbf{l} \ , \tag{59}$$

$$1 = m''_0 = m'_0 \ m_0 + \mathbf{m}' \ \mathbf{m} - l'_0 \ n_0 + \mathbf{l}' \ \mathbf{n} \ , \tag{60}$$

$$0 = \mathbf{m}'' = m'_0 \ \mathbf{m} + \mathbf{m}' \ m_0 - i \ \mathbf{m}' \times \mathbf{m} - l'_0 \ \mathbf{n} + \mathbf{l}' \ n_0 - i \ \mathbf{l}' \times \mathbf{n} \ , \tag{61}$$

$$0 = n''_0 = k'_0 \ n_0 - \mathbf{k}' \ \mathbf{n} + n'_0 \ m_0 + \mathbf{n}' \ \mathbf{m} \ , \tag{62}$$

$$0 = \mathbf{n}'' = k'_0 \ \mathbf{n} - \mathbf{k}' \ n_0 + i \ \mathbf{k}' \times \mathbf{n} + n'_0 \ \mathbf{m} + \mathbf{n}' \ m_0 - i \ \mathbf{n}' \times \mathbf{m} \ , \tag{63}$$

$$0 = l''_0 = l'_0 \ k_0 + \mathbf{l}' \ \mathbf{k} + m'_0 \ l_0 - \mathbf{m}' \ \mathbf{l} \ , \tag{64}$$

$$0 = \mathbf{l}'' = l'_0 \ \mathbf{k} + \mathbf{l}' \ k_0 + i \ \mathbf{l}' \times \mathbf{k} + m'_0 \ \mathbf{l} - \mathbf{m}' \ l_0 - i \ \mathbf{m}' \times \mathbf{l} \ . \tag{65}$$

Consider eq. (58):

$$
\begin{aligned}
1 = k'_0 \ k_0 - n'_0 \ l_0 + \mathbf{k}' \ \mathbf{k} + \mathbf{n}' \ \mathbf{l} = (\det G)^{-1} \\
\times [ \ k_0^2(mm) + m_0k_0(ln) + l_0k_0(nm) - n_0k_0(lm) + ik_0\mathbf{l}(\mathbf{m} \times \mathbf{n}) \\
+k_0l_0(nm) - m_0l_0(kn) + l_0^2(nn) + n_0l_0(km) - il_0\mathbf{k}(\mathbf{m} \times \mathbf{n}) \\
-\mathbf{k}^2(mm) - \mathbf{km}(ln) - \mathbf{lk}(nm) + \mathbf{nk}(lm) + 2\mathbf{k}(\mathbf{l} \times (\mathbf{n} \times \mathbf{m})) \\
+in_0\mathbf{k}(\mathbf{l} \times \mathbf{m}) + il_0\mathbf{k}(\mathbf{n} \times \mathbf{m}) + im_0\mathbf{k}(\mathbf{n} \times \mathbf{l}) \\
-\mathbf{kl}(nm) + \mathbf{ml}(kn) - \mathbf{l}^2(nn) - \mathbf{nl}(km) + 2\mathbf{l}(\mathbf{k} \times (\mathbf{m} \times \mathbf{n})) \\
+ik_0\mathbf{l}(\mathbf{m} \times \mathbf{n}) + im_0\mathbf{l}(\mathbf{k} \times \mathbf{n}) + in_0\mathbf{l}(\mathbf{m} \times \mathbf{k}) \ ] \ .
\end{aligned}
$$

After transformations it gives

$$
\begin{aligned}
\det G = (kk) \ (mm) + (ll) \ (nn) + 2 \ (mk) \ (ln) + 2 \ (lk) \ (nm) - 2 \ (nk) \ (lm) \\
+ 2 \ i \ [ \ k_0 \ \mathbf{l}(\mathbf{m} \times \mathbf{n}) \ + \ m_0 \ \mathbf{k}(\mathbf{n} \times \mathbf{l}) \ + \ l_0 \ \mathbf{k}(\mathbf{n} \times \mathbf{m}) \ + \ n_0 \ \mathbf{l}(\mathbf{m} \times \mathbf{k}) \ ] \\
+ 2 \ \mathbf{k} \ (\mathbf{l} \times (\mathbf{n} \times \mathbf{m})) \ - \ 2 \ \mathbf{l} \ (\mathbf{k} \times (\mathbf{n} \times \mathbf{m})) \ .
\end{aligned} \tag{66}
$$

Taking in mind identity

$$2 \ [ \ \ \mathbf{k} \ (\mathbf{l} \times (\mathbf{n} \times \mathbf{m})) \ - \ \mathbf{l} \ (\mathbf{k} \times (\mathbf{n} \times \mathbf{m})) \ \ ] = 4(\mathbf{kn}) \ (\mathbf{ml}) - 4(\mathbf{km}) \ (\mathbf{nl})$$

for $\det G$ we have

$$
\begin{aligned}
\det G = (kk) \ (mm) + (ll) \ (nn) + 2 \ (mk) \ (ln) + 2 \ (lk) \ (nm) - 2 \ (nk) \ (lm) \\
+ 2 \ i \ [ \ k_0 \ \mathbf{l}(\mathbf{m} \times \mathbf{n}) \ + \ m_0 \ \mathbf{k}(\mathbf{n} \times \mathbf{l}) \ + \ l_0 \ \mathbf{k}(\mathbf{n} \times \mathbf{m}) \ + \ n_0 \ \mathbf{l}(\mathbf{m} \times \mathbf{k}) \ ] \\
+4(\mathbf{kn}) \ (\mathbf{ml}) - 4(\mathbf{km}) \ (\mathbf{nl}) \ .
\end{aligned} \tag{67}
$$

Now, let us verify that eq. (60) leads us to the same det $G$. Indeed, from (60) it follows

$$
\begin{aligned}
1 = m_0' \, m_0 + \mathbf{m}' \, \mathbf{m} - l_0' \, n_0 + \mathbf{l}' \, \mathbf{n} = (\det G)^{-1} \\
\times [ \, k_0 m_0 \, (ln) + m_0^2 \, (kk) - l_0 m_0 \, (kn) + n_0 m_0 \, (lk) + i m_0 \, \mathbf{n}(\mathbf{l} \times \mathbf{k}) \\
-k_0 n_0 \, (ml) + m_0 n_0 \, (kl) + l_0 n_0 \, (km) + n_0^2 \, (ll) - i n_0 \, \mathbf{m}(\mathbf{l} \times \mathbf{k}) \\
-\mathbf{km} \, (ln) - \mathbf{m}^2 \, (kk) + \mathbf{l} \, \mathbf{m}(kn) - \mathbf{nm} \, (lk) + 2\mathbf{m} \, (\mathbf{n} \times (\mathbf{l} \times \mathbf{k})) \\
+ i n_0 \, \mathbf{m}(\mathbf{k} \times \mathbf{l}) + i l_0 \, \mathbf{m}(\mathbf{k} \times \mathbf{n}) + i k_0 \, \mathbf{m}(\mathbf{n} \times \mathbf{l}) \\
+\mathbf{kn} \, (ml) - \mathbf{mn} \, (kl) - \mathbf{ln} \, (km) - \mathbf{n}^2 \, (ll) + 2\mathbf{n} \, (\mathbf{m} \times (\mathbf{k} \times \mathbf{l})) \\
+ i m_0 \, \mathbf{n}(\mathbf{l} \times \mathbf{k}) + i k_0 \, \mathbf{n}(\mathbf{l} \times \mathbf{m}) + i l_0 \, \mathbf{n}(\mathbf{m} \times \mathbf{k}) \, ] \, ,
\end{aligned}
$$

from where we arrive at

$$
\begin{aligned}
\det G = (kk) \, (mm) + (ll) \, (nn) + 2 \, (mk) \, (ln) + 2 \, (lk) \, (nm) - 2 \, (nk) \, (lm) \\
+ 2 \, i \, [ \, k_0 \, \mathbf{m}(\mathbf{n} \times \mathbf{l}) \; + \; m_0 \, \mathbf{n}(\mathbf{l} \times \mathbf{k}) \; + \; l_0 \, \mathbf{m}(\mathbf{k} \times \mathbf{n}) \; + \; n_0 \, \mathbf{m}(\mathbf{k} \times \mathbf{l}) \, ] \\
+ 2 \, \mathbf{m} \, (\mathbf{n} \times (\mathbf{l} \times \mathbf{k})) \; + \; 2 \, \mathbf{n} \, (\mathbf{m} \times (\mathbf{k} \times \mathbf{l})) \, .
\end{aligned} \tag{68}
$$

Taking in mind identity

$$
2 \, \mathbf{m} \, (\mathbf{n} \times (\mathbf{l} \times \mathbf{k})) \; + \; 2 \, \mathbf{n} \, (\mathbf{m} \times (\mathbf{k} \times \mathbf{l})) = 4(\mathbf{kn}) \, (\mathbf{ml}) - 4(\mathbf{km}) \, (\mathbf{nl}) \, ;
$$

we see that eq. (68) coincides with (67).

Now, let us turn to relations (62) and (64). Firstly, consider eq. (62):

$$
\begin{aligned}
0 = k_0' \, n_0 + n_0' \, m_0 - \mathbf{k}' \, \mathbf{n} + \mathbf{n}' \, \mathbf{m} \\
= k_0 n_0 \, (m_0^2 - \mathbf{m}^2) + m_0 n_0 \, (l_0 n_0 - \mathbf{ln}) + l_0 n_0 \, (n_0 m_0 - \mathbf{nm}) \\
-n_0^2 (l_0 m_0 - \mathbf{lm}) + i n_0 \mathbf{l}(\mathbf{m} \times \mathbf{n}) \\
-k_0 m_0 \, (n_0 m_0 - \mathbf{nm}) + m_0^2 \, (k_0 n_0 - \mathbf{kn}) - l_0 m_0 \, (n_0^2 - \mathbf{n}^2) \\
-n_0 m_0 (k_0 m_0 - \mathbf{km}) + i m_0 \mathbf{k}(\mathbf{m} \times \mathbf{n}) \\
+\mathbf{kn} \, (m_0^2 - \mathbf{m}^2) + \mathbf{mn} \, (l_0 n_0 - \mathbf{ln}) + \mathbf{ln} \, (n_0 m_0 - \mathbf{nm}) - \mathbf{n}^2 \, (l_0 m_0 - \mathbf{lm}) \\
-2\mathbf{n} \, (\mathbf{l} \times (\mathbf{n} \times \mathbf{m})) - i n_0 \, \mathbf{n} \, (\mathbf{l} \times \mathbf{m}) - i l_0 \, \mathbf{n} \, (\mathbf{n} \times \mathbf{m}) - i m_0 \, \mathbf{n} \, (\mathbf{n} \times \mathbf{l}) \\
-\mathbf{km} \, (n_0 m_0 - \mathbf{nm}) + \mathbf{m}^2 \, (k_0 n_0 - \mathbf{kn}) - \mathbf{lm} \, (n_0^2 - \mathbf{n}^2) - \mathbf{nm} \, (k_0 m_0 - \mathbf{km}) \\
+2\mathbf{m} \, (\mathbf{k} \times (\mathbf{m} \times \mathbf{n})) + i k_0 \, \mathbf{m} \, (\mathbf{m} \times \mathbf{n}) + i m_0 \, \mathbf{m} \, (\mathbf{k} \times \mathbf{n}) + i n_0 \, \mathbf{m} \, (\mathbf{m} \times \mathbf{k}).
\end{aligned} \tag{69}
$$

One may verify that all terms with zero-index cancel out each other, so that we have

$$
\begin{aligned}
0 = +2\mathbf{n}^2 \, (\mathbf{lm}) + 2 \, (\mathbf{km} \, (\mathbf{nm}) - 2( \, \mathbf{kn}) \, (\mathbf{m}^2) - 2 \, (\mathbf{ln}) \, (\mathbf{nm}) \\
-2 \, \mathbf{n} \, (\mathbf{l} \times (\mathbf{n} \times \mathbf{m})) + 2 \, \mathbf{m} \, (\mathbf{k} \times (\mathbf{m} \times \mathbf{n})) \, .
\end{aligned}
$$

The later is equivalent to the identity $0 = 0$:

$$
\begin{aligned}
+ \, 2\mathbf{n}^2 \, (\mathbf{lm}) + 2 \, (\mathbf{km} \, (\mathbf{nm}) - 2( \, \mathbf{kn}) \, (\mathbf{m}^2) - 2 \, (\mathbf{ln}) \, (\mathbf{nm}) \\
-2\mathbf{n} \, [\mathbf{n} \, (\mathbf{lm}) - \mathbf{m} \, (\mathbf{ln})] + 2\mathbf{m} \, [\mathbf{m} \, (\mathbf{kn}) - \mathbf{n} \, (\mathbf{km})] \equiv 0 \, .
\end{aligned} \tag{70}
$$

In the same manner consider eq. (64):

$$
\begin{aligned}
0 &= l_0' \, k_0 + m_0' \, l_0 + \mathbf{l}' \, \mathbf{k} - \mathbf{m}' \, \mathbf{l} \\
&= k_0^2 \, (m_0 l_0 - \mathbf{ml}) - m_0 k_0 \, (k_0 l_0 - \mathbf{kl}) - l_0 k_0 \, (k_0 m_0 - \mathbf{km}) - n_0 k_0 (l_0^2 - \mathbf{l}^2) + ikn_0 \mathbf{m}(\mathbf{l} \times \mathbf{k}) \\
&\quad + k_0 l_0 \, (l_0 n_0 - \mathbf{ln}) + m_0 l_0 \, (k_0^2 - \mathbf{k}^2) - l_0^2 \, (k_0 n_0 - \mathbf{kn}) + n_0 l_0 (l_0 k_0 - \mathbf{lk}) + il_0 \mathbf{n}(\mathbf{l} \times \mathbf{k}) \\
&\quad + \mathbf{k}^2 \, (m_0 l_0 - \mathbf{ml}) - \mathbf{mk} \, (k_0 l_0 - \mathbf{kl}) - \mathbf{lk} \, (k_0 m_0 - \mathbf{km}) - \mathbf{nk} \, (l_0^2 - \mathbf{l}^2) \\
&\quad + 2\mathbf{k} \, (\mathbf{m} \times (\mathbf{k} \times \mathbf{l})) + im_0 \, \mathbf{k} \, (\mathbf{l} \times \mathbf{k}) + ik_0 \, \mathbf{k} \, (\mathbf{l} \times \mathbf{m}) + il_0 \, \mathbf{k} \, (\mathbf{m} \times \mathbf{k}) \\
&\quad + \mathbf{kl} \, (l_0 n_0 - \mathbf{ln}) + \mathbf{ml} \, (k_0^2 - \mathbf{k}^2) - \mathbf{ll} \, (k_0 n_0 - \mathbf{kn}) + \mathbf{nl} \, (l_0 k_0 - \mathbf{lk}) \\
&\quad - 2\mathbf{l} \, (\mathbf{n} \times (\mathbf{l} \times \mathbf{k})) - in_0 \, \mathbf{l} \, (\mathbf{k} \times \mathbf{l}) - il_0 \, \mathbf{l} \, (\mathbf{k} \times \mathbf{n}) - ik_0 \, \mathbf{l} \, (\mathbf{n} \times \mathbf{l}) \, , \qquad (71)
\end{aligned}
$$

and further

$$
\begin{aligned}
0 = -2\mathbf{k}^2 \, (\mathbf{ml}) + 2 \, (\mathbf{mk} \, (\mathbf{kl}) + 2( \, \mathbf{nk}) \, (\mathbf{l}^2) - 2 \, (\mathbf{kl}) \, (\mathbf{ln}) \\
+ 2 \, \mathbf{k} \, (\mathbf{m} \times (\mathbf{k} \times \mathbf{l})) - 2 \, \mathbf{l} \, (\mathbf{n} \times (\mathbf{l} \times \mathbf{k})) \, ;
\end{aligned}
$$

which is equivalent to the identity $0 = 0$:

$$
\begin{aligned}
- 2 \, \mathbf{k}^2 \, (\mathbf{ml}) + 2 \, (\mathbf{mk} \, (\mathbf{kl}) + 2 \, ( \, \mathbf{nk}) \, (\mathbf{l}^2) - 2 \, (\mathbf{kl}) \, (\mathbf{ln}) \\
+ 2 \, \mathbf{k} \, [ \, \mathbf{k} \, (\mathbf{ml}) - \mathbf{l} \, (\mathbf{mk}) \, ] - 2 \, \mathbf{l} \, [ \, \mathbf{l} \, (\mathbf{nk}) - \mathbf{k} \, (\mathbf{nl}) \, ] \equiv 0 \, . \qquad (72)
\end{aligned}
$$

Equations (59), (61),(63),(65) can be verified as well.

In the end of this section let us write down expression for $\det G$:

$$
\begin{aligned}
\det G = (kk) \, (mm) + (ll) \, (nn) + 2 \, (mk) \, (ln) + 2 \, (lk) \, (nm) - 2 \, (nk) \, (lm) + \\
+ \, 2 \, i \, [ \, k_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{n}) \; + \; m_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{l}) \; + \; l_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{m}) \; + \; n_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{k}) \, ] \\
+ 4(\mathbf{kn}) \, (\mathbf{ml}) - 4(\mathbf{km}) \, (\mathbf{nl}) \, . \qquad (73)
\end{aligned}
$$

The expression become more simple in special cases.

**Variant A**

All component with 0-index are real, all component with index 1,2,3 are imaginary. Performing the change

$$
\mathbf{k} \Longrightarrow i \, \mathbf{k}, \qquad \mathbf{m} \Longrightarrow i \, \mathbf{m}, \qquad \mathbf{l} \Longrightarrow i \, \mathbf{l}, \qquad \mathbf{n} \Longrightarrow i \, \mathbf{n}, \qquad (74)
$$

from (73) we get (the notation is used: $[ab] = a_0 a_0 + a_1 a_1 + a_2 a_2 + a_3 a_3$):

$$
\begin{aligned}
\det G = [kk] \, [mm] + [ll] \, [nn] + 2 \, [mk] \, [ln] + 2 \, [lk] \, [nm] - 2 \, [nk] \, [lm]) \\
+ \, 2 \; [ \, k_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{n}) \; + \; m_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{l}) \; + \; l_0 \, \mathbf{k}(\mathbf{n} \times \mathbf{m}) \; + \; n_0 \, \mathbf{l}(\mathbf{m} \times \mathbf{k}) \, ] \\
+ 4(\mathbf{kn}) \, (\mathbf{ml}) - 4(\mathbf{km}) \, (\mathbf{nl}) \, , \qquad (75)
\end{aligned}
$$

here all the quantities are real-valued.

**Variant B**

Restrictions imposed are

$$m_a = k_a^*, \qquad l_a = n_a^*, \tag{76}$$

and from (73) it follows

$$\det G = (kk)\,(kk)^* + (nn)^*\,(nn) + 2\,(k^*k)\,(n^*n) + 2\,(n^*k)\,(nk^*) - 2\,(nk)\,(nk)^*$$
$$+ 2\,i\,[\,k_0\,\mathbf{n}^*(\mathbf{k}^* \times \mathbf{n}) \;+\; k_0^*\,\mathbf{k}(\mathbf{n} \times \mathbf{n}^*) \;+\; n_0^*\,\mathbf{k}(\mathbf{n} \times \mathbf{k}^*) \;+\; n_0\,\mathbf{n}^*(\mathbf{k}^* \times \mathbf{k})\,]$$
$$+4(\mathbf{kn})\,(\mathbf{k}^*\mathbf{n}^*) - 4(\mathbf{kk}^*)\,(\mathbf{nn}^*)\,. \tag{77}$$

The latter can be rewritten in the form

$$\det G = (kk)\,(kk)^* + (nn)^*\,(nn) + 2\,(k^*k)\,(n^*n) + 2\,(n^*k)\,(nk^*) - 2\,(nk)\,(nk)^*$$
$$+ 2\,i\,[\,k_0\,\mathbf{k}^*(\mathbf{n} \times \mathbf{n}^*) \;-\; k_0^*\,\mathbf{k}(\mathbf{n}^* \times \mathbf{n}) \;+\; n_0^*\,\mathbf{n}(\mathbf{k} \times \mathbf{k}^*) \;-\; n_0\,\mathbf{n}^*(\mathbf{k}^* \times \mathbf{k})\,]$$
$$+4(\mathbf{kn})\,(\mathbf{k}^*\mathbf{n}^*) - 4(\mathbf{kk}^*)\,(\mathbf{nn}^*)\,, \tag{78}$$

which is explicitly real-valued.

**Variant C**

In formulas (75) one should take additional restrictions:

$$m_0 = +k_0, \qquad l_0 = n_0, \qquad \mathbf{m} = -\mathbf{k}, \qquad \mathbf{l} = -\mathbf{n}\,; \tag{79}$$

$$\det G = [kk]\,[kk] + [nn]\,[nn] + 2\,(kk)\,(nn) + 2\,(nk)\,(nk) - 2\,[nk]\,[nk])$$
$$+ 2\;\;[\,k_0\,\mathbf{n}(\mathbf{k} \times \mathbf{n}) \;-\; k_0\,\mathbf{k}(\mathbf{n} \times \mathbf{n}) \;-\; n_0\,\mathbf{k}(\mathbf{n} \times \mathbf{k}) \;+\; n_0\,\mathbf{n}(\mathbf{k} \times \mathbf{k})\,]$$
$$+4(\mathbf{kn})\,(\mathbf{kn}) - 4(\mathbf{kk})\,(\mathbf{nn})\,,$$

and further

$$\det G = [kk]\,[kk] + [nn]\,[nn]$$
$$+ 2\,(kk)\,(nn) + 2\,(nk)\,(nk) - 2\,[nk]\,[nk]) + 4(\mathbf{kn})\,(\mathbf{kn}) - 4(\mathbf{kk})\,(\mathbf{nn})\,. \tag{80}$$

## 5 Independent calculation of det $G$

Starting from the form

$$G = \left| \begin{array}{rrrr} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{array} \right| \tag{81}$$

let us calculate $\det G$ by direct method of linear algebra:

$$\det G = \begin{vmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{vmatrix}$$

$$= G_{11} \begin{vmatrix} G_{22} & G_{23} & G_{24} \\ G_{32} & G_{33} & G_{34} \\ G_{42} & G_{43} & G_{44} \end{vmatrix} - G_{12} \begin{vmatrix} G_{21} & G_{23} & G_{24} \\ G_{31} & G_{33} & G_{34} \\ G_{41} & G_{43} & G_{44} \end{vmatrix} +$$

$$+ G_{13} \begin{vmatrix} G_{21} & G_{22} & G_{24} \\ G_{31} & G_{32} & G_{34} \\ G_{41} & G_{42} & G_{44} \end{vmatrix} - G_{14} \begin{vmatrix} G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \\ G_{41} & G_{42} & G_{43} \end{vmatrix}$$

and further

$$\det G$$

$$= G_{11} \; ( \; G_{22} \begin{vmatrix} G_{33} & G_{34} \\ G_{43} & G_{44} \end{vmatrix} - G_{23} \begin{vmatrix} G_{32} & G_{34} \\ G_{42} & G_{44} \end{vmatrix} + G_{24} \begin{vmatrix} G_{32} & G_{33} \\ G_{42} & G_{43} \end{vmatrix} )$$

$$- G_{12} \; ( \; G_{21} \begin{vmatrix} G_{33} & G_{34} \\ G_{43} & G_{44} \end{vmatrix} - G_{23} \begin{vmatrix} G_{31} & G_{34} \\ G_{41} & G_{44} \end{vmatrix} + G_{24} \begin{vmatrix} G_{31} & G_{33} \\ G_{41} & G_{43} \end{vmatrix} )$$

$$+ G_{13} \; ( \; G_{21} \begin{vmatrix} G_{32} & G_{34} \\ G_{42} & G_{44} \end{vmatrix} - G_{22} \begin{vmatrix} G_{31} & G_{34} \\ G_{41} & G_{44} \end{vmatrix} + G_{24} \begin{vmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{vmatrix} )$$

$$- G_{14} \; ( \; G_{21} \begin{vmatrix} G_{32} & G_{33} \\ G_{42} & G_{43} \end{vmatrix} - G_{22} \begin{vmatrix} G_{31} & G_{33} \\ G_{41} & G_{43} \end{vmatrix} + G_{23} \begin{vmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{vmatrix} ) \, .$$

From this it follows

$$\det G =$$

$$= (G_{11}\, G_{22} - G_{12}\, G_{21}) \begin{vmatrix} G_{33} & G_{34} \\ G_{43} & G_{44} \end{vmatrix} + (-G_{11}\, G_{23} + G_{13}\, G_{21}) \begin{vmatrix} G_{32} & G_{34} \\ G_{42} & G_{44} \end{vmatrix}$$

$$+ (G_{11}\, G_{24} - G_{14}\, G_{21}) \begin{vmatrix} G_{32} & G_{33} \\ G_{42} & G_{43} \end{vmatrix} + (G_{12}\, G_{23} - G_{13}\, G_{22}) \begin{vmatrix} G_{31} & G_{34} \\ G_{41} & G_{44} \end{vmatrix}$$

$$+ (-G_{12}\, G_{24} + G_{14}\, G_{22}) \begin{vmatrix} G_{31} & G_{33} \\ G_{41} & G_{43} \end{vmatrix} + (G_{13}\, G_{24} - G_{14}\, G_{23}) \begin{vmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{vmatrix} .$$

Allowing for

$$(G_{kl}) = \begin{vmatrix} +(k_0 + k_3) & +(k_1 - ik_2) & +(n_0 - n_3) & -(n_1 - in_2) \\ +(k_1 + ik_2) & +(k_0 - k_3) & -(n_1 + in_2) & +(n_0 + n_3) \\ -(l_0 + l_3) & -(l_1 - il_2) & +(m_0 - m_3) & -(m_1 - im_2) \\ -(l_1 + il_2) & -(l_0 - l_3) & -(m_1 + im_2) & +(m_0 + m_3) \end{vmatrix} ,$$

we arrive at the following form for $\det G$:

$$
\begin{aligned}
&\det G = \\
&[(k_0+k_3)(k_0-k_3)-(k_1-ik_2)(k_1+ik_2)][(m_0-m_3)(m_0+m_3)-(m_1-im_2)(m_1+im_2)] \\
&+[(k_0+k_3)(n_1+in_2)+(n_0-n_3)(k_1+ik_2)][-(l_1-il_2)(m_0+m_3)-(m_1-im_2)(l_0-l_3)] \\
&+[(k_0+k_3)(n_0+n_3)+(n_1-in_2)(k_1+ik_2)][(l_1-il_2)(m_1+im_2)+(m_0-m_3)(l_0-l_3)] \\
&+[-(k_1-ik_2)(n_1+in_2)-(n_0-n_3)(k_0-k_3)][-(l_0+l_3)(m_0+m_3)-(m_1-im_2)(l_1+il_2)] \\
&+[-(k_1-ik_2)(n_0+n_3)-(n_1-in_2)(k_0-k_3)][(l_0+l_3)(m_1+im_2)+(m_0-m_3)(l_1+il_2)] \\
&+[(n_0-n_3)(n_0+n_3)-(n_1-in_2)(n_1+in_2)][(l_0+l_3)(l_0-l_3)-(l_1-il_2)(l_1+il_2)] \,.
\end{aligned} \tag{82}
$$

and further (1-th and 6-th rows give simple terms)

$$
\begin{aligned}
&\det G = (k_0^2-k_1^2-k_2^2-k_3^2)(m_0^2-m_1^2-m_2^2-m_3^2)+(n_0^2-n_1^2-n_2^2-n_3^2)(l_0^2-l_1^2-l_2^2-l_3^2) \\
&+[(k_0+k_3)(n_1+in_2)+(n_0-n_3)(k_1+ik_2)][-(l_1-il_2)(m_0+m_3)-(m_1-im_2)(l_0-l_3)] \\
&+[(k_0+k_3)(n_0+n_3)+(n_1-in_2)(k_1+ik_2)][\,(l_1-il_2)(m_1+im_2)+(m_0-m_3)(l_0-l_3)]+ \\
&[-(k_1-ik_2)(n_1+in_2)-(n_0-n_3)(k_0-k_3)\,][-(l_0+l_3)(m_0+m_3)-(m_1-im_2)(l_1+il_2)] \\
&+[-(k_1-ik_2)(n_0+n_3)-(n_1-in_2)(k_0-k_3)][(l_0+l_3)(m_1+im_2)+(m_0-m_3)(l_1+il_2)].
\end{aligned} \tag{83}
$$

Consider four separate terms:

$$
\begin{aligned}
(1) &= [\ (k_0+k_3)\ (n_1+in_2)+(n_0-n_3)\ (k_1+ik_2)\ ] \\
&\times[\ -(l_1-il_2)\ (m_0+m_3)-(m_1-im_2)\ (l_0-l_3)\ ] \\
&= [\ (k_0n_1+k_1n_0)+(k_3n_1-k_1n_3)+i\ (k_0n_2+k_2n_0)+i\ (k_3n_2-k_2n_3)\ ] \\
&\times[\ -(m_0l_1+m_1l_0)+(m_1l_3-m_3l_1)+i\ (m_0l_2+m_2l_0)+i\ (m_3l_2-m_2l_3)\ ]\ ;
\end{aligned}
$$

$$
\begin{aligned}
(2) &= [\ (k_0+k_3)\ (n_0+n_3)+(n_1-in_2)\ (k_1+ik_2)\ ] \\
&\times\ [\ (l_1-il_2)\ (m_1+im_2)+(m_0-m_3)\ (l_0-l_3)\ ] \\
&= [\ (k_0n_3+k_3n_0)+k_0n_0+(n_1k_1+n_2k_2+k_3n_3)+i\ (n_1k_2-n_2k_1)] \\
&\times\ [\ -(m_0l_3+m_3l_0)+m_0l_0+(l_1m_1+l_2m_2+m_3l_3)\ +i\ (l_1m_2-l_2m_1)\ ]\ ;
\end{aligned}
$$

$$
\begin{aligned}
(3) &= [\ -(k_1-ik_2)\ (n_1+in_2)-(n_0-n_3)\ (k_0-k_3)\ ] \\
&\times\ [\ -(l_0+l_3)\ (m_0+m_3)-(m_1-im_2)(l_1+il_2)\ ] \\
&[\ (n_0k_3+n_3k_0)-n_0k_0-(k_1n_1+k_2n_2+n_3k_3)+i\ (k_2n_1-k_1n_2)] \\
&\times\ [\ -(l_0m_3+l_3m_0)-l_0m_0-(l_1m_1+l_2m_2+l_3m_3)+i\ (l_1m_2-l_2m_1)\ ]\ ;
\end{aligned}
$$

$$
\begin{aligned}
(4) &= [\ -(k_1-ik_2)\ (n_0+n_3)-(n_1-in_2)\ (k_0-k_3)\ ] \\
&\times\ [\ (l_0+l_3)\ (m_1+im_2)+(m_0-m_3)\ (l_1+il_2)\ ] \\
&= [\ -(k_1n_0+k_0n_1)+(n_1k_3-k_1n_3)+i\ (k_2n_0+n_2k_0)+i\ (k_2n_3-n_2k_3)\ ] \\
&\times\ [\ (l_0m_1+l_1m_0)+(l_3m_1-l_1m_3)+i\ (l_0m_2+l_2m_0)+i\ (l_3m_2-l_2m_3)\ ]\ .
\end{aligned}
$$

It is convenient to introduce the notation:

$$
\mathbf{k}\times\mathbf{n}=\mathbf{A}\ , \qquad \mathbf{m}\times\mathbf{l}=\mathbf{B}\ , \tag{84}
$$

then previous formulas look shorter

$$\det G = (\ (kk)\ (mm) + (nn)\ (ll) + (1) + (2) + (3) = (4)\ ,$$

$$(1) = [\ (k_0 n_1 + k_1 n_0) + A_2 + i\ (k_0 n_2 + k_2 n_0) - i\ A_1\ ]$$
$$\times [\ -(m_0 l_1 + m_1 l_0) - B_2 + i\ (m_0 l_2 + m_2 l_0) - iB_1\ ]\ ,$$

$$(2) = [\ (k_0 n_3 + k_3 n_0) + k_0 n_0 + \mathbf{nk} - iA_3\ ]$$
$$\times [\ -(m_0 l_3 + m_3 l_0) + m_0 l_0 + \mathbf{lm} - iB_3\ ]\ ,$$

$$(3) = [\ (n_0 k_3 + n_3 k_0) - n_0 k_0 - \mathbf{kn} - i\ A_3\ ]$$
$$\times [\ -l_0 m_0 - (l_0 m_3 + l_3 m_0) - \mathbf{lm} - iB_3\ ]\ ,$$

$$(4) = [\ -(k_1 n_0 + k_0 n_1) + A_2 + i\ (k_2 n_0 + n_2 k_0) + iA_1\ ]$$
$$\times [\ (l_0 m_1 + l_1 m_0) - B_2 + i\ (l_0 m_2 + l_2 m_0) + iB_1\ ]\ .$$

With the use of simplifying notation

$$k_0\ n_i = N_i\ , \qquad m_0\ l_i = L_i\ , \qquad n_0\ k_i = K_i\ , \qquad l_0\ m_i = M_i\ , \tag{85}$$

previous formulas look simpler:

$$\begin{aligned}
(1) &= [\ (N_1 + K_1) + A_2 + i\ (N_2 + K_2) - i\ A_1\ ] \\
&\times [\ -(L_1 + M_1) - B_2 + i\ (L_2 + M_2) - iB_1\ ]\ , \\
(2) &= [\ (N_3 + K_3) + k_0 n_0 + \mathbf{nk} - iA_3\ ] \\
&\times [\ -(L_3 + M_3) + m_0 l_0 + \mathbf{lm} - iB_3\ ]\ ; \\
(3) &= [\ (K_3 + N_3) - n_0 k_0 - \mathbf{kn} - i\ A_3\ ] \\
&\times [\ -l_0 m_0 - (M_3 + L_3) - \mathbf{lm} - iB_3\ ]\ , \\
(4) &= [\ -(K_1 + N_1) + A_2 + i\ (K_2 + N_2) + iA_1\ ] \\
&\times [\ (M_1 + L_1) - B_2 + i\ (M_2 + L_2) + iB_1\ ]\ .
\end{aligned} \tag{86}$$

Terms (1)-(4) in explicit form are

$$\begin{aligned}
(1) = &-N_1 L_1 - N_1 M_1 - N_1 B_2 + i\ N_1 L_2 + iN_1 M_2 - iN_1 B_1 \\
&-K_1 L_1 - K_1 M_1 - K_1 B_2 + iK_1 L_2 + iK_1 M_2 - iK_1 B_1 \\
&-A_2 L_1 - A_2 M_1 - A_2 B_2 + iA_2 L_2 + iA_2 M_2 - iA_2 B_1 \\
&-iN_2 L_1 - iN_2 M_1 - iN_2 B_2 - N_2 L_2 - N_2 M_2 + N_2 B_1 - \\
&-iK_2 L_1 - iK_2 M_1 - iK_2 B_2 - K_2 L_2 - K_2 M_2 + K_2 B_1 \\
&+iA_1 L_1 + iA_1 M_1 + iA_1 B_2 + A_1 L_2 + A_1 M_2 - A_1 B_1\ ,
\end{aligned}$$

$$\begin{aligned}
(2) = &-N_3 L_3 - N_3 M_3 + N_3 m_0 l_0 + N_3 \mathbf{lm} - iN_3 B_3 \\
&-K_3 L_3 - K_3 M_3 + K_3 m_0 l_0 + K_3 \mathbf{lm} - iK_3 B_3\ - \\
&-k_0 n_0 L_3 - k_0 n_0 M_3 + k_0 n_0 m_0 l_0 + k_0 n_0 \mathbf{lm} - ik_0 n_0 B_3 \\
&-(\mathbf{nk}) L_3 - (\mathbf{nk}) M_3 + (\mathbf{nk}) m_0 l_0 + (\mathbf{nk})(\mathbf{lm}) - i(\mathbf{nk}) B_3 \\
&+iA_3 L_3 + iA_3 M_3 - iA_3 m_0 l_0 - iA_3 \mathbf{lm} - A_3 B_3\ ,
\end{aligned}$$

$$
\begin{aligned}
(3) = &-K_3 l_0 m_0 - K_3 M_3 - K_3 L_3 - K_3(\mathbf{lm}) - iK_3 B_3 \\
&-N_3 l_0 m_0 - N_3 M_3 - N_3 L_3 - N_3(\mathbf{lm}) - iN_3 B_3 \\
&+n_0 k_0 l_0 m_0 + n_0 k_0 M_3 + n_0 k_0 L_3 + n_0 k_0(\mathbf{lm}) + in_0 k_0 B_3 \\
&+(\mathbf{kn}) l_0 m_0 + (\mathbf{kn}) M_3 + (\mathbf{kn}) L_3 + (\mathbf{kn})(\mathbf{lm}) + i(\mathbf{kn}) B_3 \\
&+i\ A_3 l_0 m_0 + i\ A_3 M_3 + i\ A_3 L_3 + i\ A_3(\mathbf{lm}) -\ A_3 B_3 \ ,
\end{aligned}
$$

$$
\begin{aligned}
(4) = &\ [\ -(K_1 + N_1) + A_2 + i\ (K_2 + N_2) + iA_1 \\
&\times\ ]\ [\ (M_1 + L_1) - B_2 + i\ (M_2 + L_2) + iB_1\ ] \\
= &-K_1 M_1 - K_1 L_1 + K_1 B_2 - iK_1 M_2 - iK_1 L_2 - iK_1 B_1 \\
&-N_1 M_1 - N_1 L_1 + N_1 B_2 - iN_1 M_2 - N_1 i L_2 - iN_1 B_1 \\
&+A_2 M_1 + A_2 L_1 - A_2 B_2 + iA_2 M_2 + iA_2 L_2 + iA_2 B_1 \\
&+iK_2 M_1 + iK_2 L_1 - iK_2 B_2 - K_2 M_2 - K_2 L_2 - K_2 B_1 \\
&+iN_2 M_1 + iN_2 L_1 - iN_2 B_2 - N_2 M_2 - N_2 L_2 - N_2 B_1 \\
&+iA_1 M_1 + iA_1 L_1 - iA_1 B_2 - A_1 M_2 - A_1 L_2 - A_1 B_1 \ .
\end{aligned}
$$

Summing these four relations:

$$
\begin{aligned}
&(1) + (2) + (3) + (4) \\
= &-N_1 L_1 - N_1 M_1 - N_1 B_2 + i\ N_1 L_2 + iN_1 M_2 - iN_1 B_1 \\
&-K_1 L_1 - K_1 M_1 - K_1 B_2 + iK_1 L_2 + iK_1 M_2 - iK_1 B_1 \\
&-A_2 L_1 - A_2 M_1 - A_2 B_2 + iA_2 L_2 + iA_2 M_2 - iA_2 B_1 \\
&-iN_2 L_1 - iN_2 M_1 - iN_2 B_2 - N_2 L_2 - N_2 M_2 + N_2 B_1 - \\
&-iK_2 L_1 - iK_2 M_1 - iK_2 B_2 - K_2 L_2 - K_2 M_2 + K_2 B_1 \\
&+iA_1 L_1 + iA_1 M_1 + iA_1 B_2 + A_1 L_2 + A_1 M_2 - A_1 B_1 \\
&-N_3 L_3 - N_3 M_3 + N_3 m_0 l_0 + N_3 \mathbf{lm} - iN_3 B_3 \\
&-K_3 L_3 - K_3 M_3 + K_3 m_0 l_0 + K_3 \mathbf{lm} - iK_3 B_3 \\
&-k_0 n_0 L_3 - k_0 n_0 M_3 + k_0 n_0 m_0 l_0 + k_0 n_0 \mathbf{lm} - ik_0 n_0 B_3 \\
&-(\mathbf{nk}) L_3 - (\mathbf{nk}) M_3 + (\mathbf{nk}) m_0 l_0 + (\mathbf{nk})(\mathbf{lm}) - i(\mathbf{nk}) B_3 \\
&+iA_3 L_3 + iA_3 M_3 - iA_3 m_0 l_0 - iA_3 \mathbf{lm}\ - -A_3 B_3
\end{aligned}
$$

$$
\begin{aligned}
&- K_3 l_0 m_0 - K_3 M_3 - K_3 L_3 - K_3(\mathbf{lm}) - iK_3 B_3 \\
&-N_3 l_0 m_0 - N_3 M_3 - N_3 L_3 - N_3(\mathbf{lm}) - iN_3 B_3 \\
&+n_0 k_0 l_0 m_0 + n_0 k_0 M_3 + n_0 k_0 L_3 + n_0 k_0(\mathbf{lm}) + in_0 k_0 B_3 \\
&+(\mathbf{kn}) l_0 m_0 + (\mathbf{kn}) M_3 + (\mathbf{kn}) L_3 + (\mathbf{kn})(\mathbf{lm}) + i(\mathbf{kn}) B_3\ + \\
&+i\ A_3 l_0 m_0 + i\ A_3 M_3 + i\ A_3 L_3 + i\ A_3(\mathbf{lm}) -\ A_3 B_3 \\
&-K_1 M_1 - K_1 L_1 + K_1 B_2 - iK_1 M_2 - iK_1 L_2 - iK_1 B_1 \\
&-N_1 M_1 - N_1 L_1 + N_1 B_2 - iN_1 M_2 - N_1 i L_2 - iN_1 B_1 \\
&+A_2 M_1 + A_2 L_1 - A_2 B_2 + iA_2 M_2 + iA_2 L_2 + iA_2 B_1 \\
&+iK_2 M_1 + iK_2 L_1 - iK_2 B_2 - K_2 M_2 - K_2 L_2 - K_2 B_1 \\
&+iN_2 M_1 + iN_2 L_1 - iN_2 B_2 - N_2 M_2 - N_2 L_2 - N_2 B_1 \\
&+iA_1 M_1 + iA_1 L_1 - iA_1 B_2 - A_1 M_2 - A_1 L_2 - A_1 B_1 \ .
\end{aligned}
$$

After simple evident simplifications we get

$$\begin{aligned}
&(1)+(2)+(3)+(4)\\
&= -2\,\mathbf{NL} - 2\,\mathbf{NM} - 2\,\mathbf{KL} - 2\,\mathbf{KM} - 2\,\mathbf{AB}\\
&-2i\,\mathbf{NB} - 2i\,\mathbf{KB} + 2i\,\mathbf{AL} + 2i\,\mathbf{AM} +\\
&+N_3 m_0 l_0 + N_3(\mathbf{lm})\\
&+K_3 m_0 l_0 + K_3(\mathbf{lm})\\
&-k_0 n_0 L_3 - k_0 n_0 M_3 + k_0 n_0 m_0 l_0 + k_0 n_0 \mathbf{lm} - i k_0 n_0 B_3\\
&-(\mathbf{nk}) L_3 - (\mathbf{nk}) M_3 + (\mathbf{nk}) m_0 l_0 + (\mathbf{nk})(\mathbf{lm}) - i(\mathbf{nk}) B_3\\
&-iA_3 m_0 l_0 - iA_3(\mathbf{lm}) -\\
&-K_3 l_0 m_0 - K_3(\mathbf{lm})\\
&-N_3 l_0 m_0 - N_3(\mathbf{lm})\\
&+n_0 k_0 l_0 m_0 + n_0 k_0 M_3 + n_0 k_0 L_3 + n_0 k_0(\mathbf{lm}) + i n_0 k_0 B_3\\
&+(\mathbf{kn}) l_0 m_0 + (\mathbf{kn}) M_3 + (\mathbf{kn}) L_3 + (\mathbf{kn})(\mathbf{lm}) + i(\mathbf{kn}) B_3\\
&+i\, A_3 l_0 m_0 + i\, A_3(\mathbf{lm})\ .
\end{aligned} \tag{87}$$

Noting that all terms containing $N_3, K_3, A_3$ cancel out each other, so we get

$$\begin{aligned}
&(1)+(2)+(3)+(4)\\
&= -2\,\mathbf{NL} - 2\,\mathbf{NM} - 2\,\mathbf{KL} - 2\,\mathbf{KM} - 2\,\mathbf{AB}\\
&-2i\,\mathbf{NB} - 2i\,\mathbf{KB} + 2i\,\mathbf{AL} + 2i\,\mathbf{AM}\\
&-k_0 n_0 L_3 - k_0 n_0 M_3 + k_0 n_0 m_0 l_0 + k_0 n_0 \mathbf{lm} - i k_0 n_0 B_3\\
&-(\mathbf{nk}) L_3 - (\mathbf{nk}) M_3 + (\mathbf{nk}) m_0 l_0 + (\mathbf{nk})(\mathbf{lm}) - i(\mathbf{nk}) B_3\\
&+n_0 k_0 l_0 m_0 + n_0 k_0 M_3 + n_0 k_0 L_3 + n_0 k_0(\mathbf{lm}) + i n_0 k_0 B_3\\
&+(\mathbf{kn}) l_0 m_0 + (\mathbf{kn}) M_3 + (\mathbf{kn}) L_3 + (\mathbf{kn})(\mathbf{lm}) + i(\mathbf{kn}) B_3\ .
\end{aligned} \tag{88}$$

and further

$$\begin{aligned}
&(1)+(2)+(3)+(4)\\
&= -2\,\mathbf{NL} - 2\,\mathbf{NM} - 2\,\mathbf{KL} - 2\,\mathbf{KM} - 2\,\mathbf{AB}\\
&-2i\,\mathbf{NB} - 2i\,\mathbf{KB} + 2i\,\mathbf{AL} + 2i\,\mathbf{AM}\\
&+2\,k_0 n_0 m_0 l_0 + 2\,k_0 n_0\,(\mathbf{lm}) + 2\,(\mathbf{nk})\,m_0 l_0 + 2\,(\mathbf{nk})\,(\mathbf{lm})\ .
\end{aligned} \tag{89}$$

Therefore, determinant of $G$ is given by

$$\begin{aligned}
&\det G (kk)\,(mm) + (nn)\,(ll)\\
&-2\,\mathbf{NL} - 2\,\mathbf{NM} - 2\,\mathbf{KL} - 2\,\mathbf{KM} - 2\,\mathbf{AB}\\
&-2i\,\mathbf{NB} - 2i\,\mathbf{KB} + 2i\,\mathbf{AL} + 2i\,\mathbf{AM}\ +\\
&+2\,k_0 n_0 (m_0 l_0 + \mathbf{lm}) + 2\,(\mathbf{nk})(\,m_0 l_0 + \mathbf{lm})\ ,
\end{aligned} \tag{90}$$

or by a shorter relation

$$
\begin{aligned}
\det G = (kk)\ (mm) + (nn)\ (ll) \\
+\ 2\ (k_0 n_0 + \mathbf{nk})\ (m_0 l_0 + \mathbf{lm}) \\
-2\ \mathbf{NL} - 2\ \mathbf{NM} - 2\ \mathbf{KL} - 2\ \mathbf{KM} - 2\ \mathbf{AB} \\
-2i\ \mathbf{NB} - 2i\ \mathbf{KB} + 2i\ \mathbf{AL} + 2i\ \mathbf{AM}\ .
\end{aligned} \tag{91}
$$

The latter formulas allows further simplification:

$$
\begin{aligned}
\det G(kk)\ (mm) + (nn)\ (ll) \\
+2\ (k_0 n_0 + \mathbf{nk})\ (m_0 l_0 + \mathbf{lm}) \\
-2\ (\mathbf{N} + \mathbf{K} - i\mathbf{A})(\mathbf{L} + \mathbf{M} + i\mathbf{B})\ .
\end{aligned} \tag{92}
$$

Besides, with the notation

$$
[kn] = k_0 n_0 + \mathbf{nk}\ , \qquad [ml] = m_0 l_0 + \mathbf{lm}\ ,
$$

relation (92) can be written as

$$
\begin{aligned}
\det G = (kk)\ (mm) + (nn)\ (ll) + 2\ [kn]\ [ml] \\
-2\ (\mathbf{N} + \mathbf{K} - i\mathbf{A})(\mathbf{L} + \mathbf{M} + i\mathbf{B})\ .
\end{aligned} \tag{93}
$$

Remembering the designation

$$
\begin{aligned}
k_0\ \mathbf{n} = \mathbf{N}\ , \qquad m_0\ \mathbf{l} = \mathbf{L}\ , \\
n_0\ \mathbf{k} = \mathbf{K}\ , \qquad l_0\ \mathbf{m} = \mathbf{M}\ , \\
\mathbf{k} \times \mathbf{n} = \mathbf{A}\ , \qquad \mathbf{m} \times \mathbf{l} = \mathbf{B}\ ,
\end{aligned} \tag{94}
$$

eq. (93) take the form

$$
\begin{aligned}
\det G = (kk)\ (mm) + (nn)\ (ll) + 2\ [kn]\ [ml]\ - \\
-2\ (\ k_0\ \mathbf{n} + n_0\ \mathbf{k} - i\ \mathbf{k} \times \mathbf{n}\ )\ (\ m_0\ \mathbf{l} + l_0\ \mathbf{m} + i\ \mathbf{m} \times \mathbf{l}\ )\ .
\end{aligned} \tag{95}
$$

In turn, eq. (91) takes the form

$$
\begin{aligned}
\det G = (kk)\ (mm) + (nn)\ (ll) + 2\ (k_0 n_0 + \mathbf{nk})\ (m_0 l_0 + \mathbf{lm}) \\
-2\ k_0 m_0\ (\mathbf{nl}) - 2\ k_0 l_0\ (\mathbf{nm}) - 2\ n_0 m_0\ (\mathbf{kl}) - 2\ n_0 l_0\ (\mathbf{km}) - 2\ (\mathbf{k} \times \mathbf{n})\ (\mathbf{m} \times \mathbf{l}) \\
-2ik_0\ \mathbf{n}(\mathbf{m} \times \mathbf{l}) - 2i\ n_0 \mathbf{k}(\mathbf{m} \times \mathbf{l}) + 2im_0\ (\mathbf{k} \times \mathbf{n})\mathbf{l} + 2il_0\ (\mathbf{k} \times \mathbf{n})\mathbf{m}\ .
\end{aligned} \tag{96}
$$

Allowing for cyclic symmetry, the later can be changed to

$$
\begin{aligned}
\det G = (kk)\ (mm) + (nn)\ (ll) \\
+2\ (k_0 n_0 + \mathbf{nk})\ (m_0 l_0 + \mathbf{lm}) - 2\ k_0 m_0\ (\mathbf{nl}) - 2\ k_0 l_0\ (\mathbf{nm}) \\
-2\ n_0 m_0\ (\mathbf{kl}) - 2\ n_0 l_0\ (\mathbf{km})\ - 2\ (\mathbf{k} \times \mathbf{n})\ (\mathbf{m} \times \mathbf{l}) \\
+2i\ [\ +k_0\ \mathbf{l}(\mathbf{m} \times \mathbf{n}) + m_0\ \mathbf{k}(\mathbf{n} \times \mathbf{l}) + l_0\ \mathbf{k}(\mathbf{n} \times \mathbf{m}) +\ n_0 \mathbf{l}(\mathbf{m} \times \mathbf{k})\ ]\ .
\end{aligned} \tag{97}
$$

Now, one should compare eq. (97) with eq. (67):

$$\det G = (kk)\ (mm) + (ll)\ (nn)$$
$$+2\ (mk)\ (ln) + 2\ (lk)\ (nm) - 2\ (nk)\ (lm) + 4(\mathbf{kn})\ (\mathbf{ml}) - 4(\mathbf{km})\ (\mathbf{nl})$$
$$+\ 2\ i\ [\ k_0\ \mathbf{l}(\mathbf{m}\times\mathbf{n})\ +\ m_0\ \mathbf{k}(\mathbf{n}\times\mathbf{l})\ +\ l_0\ \mathbf{k}(\mathbf{n}\times\mathbf{m})\ +\ n_0\ \mathbf{l}(\mathbf{m}\times\mathbf{k})\ ]\ , \tag{98}$$

they are the same only if

$$2\ (k_0 n_0 + \mathbf{nk})\ (m_0 l_0 + \mathbf{lm}) - 2\ k_0 m_0\ (\mathbf{nl}) - 2\ k_0 l_0\ (\mathbf{nm})$$
$$-2\ n_0 m_0\ (\mathbf{kl}) - 2\ n_0 l_0\ (\mathbf{km}) - 2\ (\mathbf{k}\times\mathbf{n})\ (\mathbf{m}\times\mathbf{l})$$
$$= 2\ (mk)\ (ln) + 2\ (lk)\ (nm) - 2\ (nk)\ (lm)$$
$$+4\ (\mathbf{kn})\ (\mathbf{ml}) - 4\ (\mathbf{km})\ (\mathbf{nl})\ . \tag{99}$$

For the right part we have

$$\underline{\text{The right}} = 2(m_0 k_0 - \mathbf{mk})(l_0 n_0 - \mathbf{ln})$$
$$+2(l_0 k_0 - \mathbf{lk})(n_0 m_0 - \mathbf{nm}) -$$
$$-2(n_0 k_0 - \mathbf{nk})(l_0 m_0 - \mathbf{lm})$$
$$+4(\mathbf{kn})\ (\mathbf{ml}) - 4(\mathbf{km})\ (\mathbf{nl})$$
$$= 2m_0 k_0 n_0 m_0 - 2m_0 k_0(\mathbf{ln}) - 2l_0 n_0(\mathbf{mk}) + 2(\mathbf{mk})(\mathbf{ln})$$
$$+2l_0 k_0\ n_0 m_0 - 2l_0 k_0(\mathbf{nm}) - 2n_0 m_0(\mathbf{lk}) + 2(\mathbf{lk})(\mathbf{nm})$$
$$-2l_0 k_0\ n_0 m_0 + 2n_0 k_0(\mathbf{lm}) + 2l_0 m_0(\mathbf{nk}) - 2(\mathbf{nk})(\mathbf{lm})$$
$$+4(\mathbf{kn})\ (\mathbf{ml}) - 4(\mathbf{km})\ (\mathbf{nl})\ ,$$

that is

$$\underline{\text{The right}} = 2m_0 k_0 n_0 m_0 - 2m_0 k_0(\mathbf{ln}) - 2l_0 n_0(\mathbf{mk}) - 2(\mathbf{mk})(\mathbf{ln})$$
$$-2l_0 k_0(\mathbf{nm}) - 2n_0 m_0(\mathbf{lk}) + 2(\mathbf{lk})(\mathbf{nm})$$
$$+2n_0 k_0(\mathbf{lm}) + 2l_0 m_0(\mathbf{nk}) + 2(\mathbf{nk})(\mathbf{lm})\ . \tag{100}$$

Taking in mind the identity

$$-\ 2\ (\mathbf{k}\times\mathbf{n})\ (\mathbf{m}\times\mathbf{l}) = -2\ \epsilon_{abc} k_b n_c\ \epsilon_{aps} m_p l_s$$
$$= -2\ (\delta_{bp}\ \delta_{cs} - \delta_{bs}\ \delta_{cp})\ k_b n_c\ m_p l_s = -2\ (\mathbf{km})\ (\mathbf{nl}) + 2\ (\mathbf{kl})\ (\mathbf{nm})\ ,$$

for the left part

$$\underline{\text{The left}\ } = 2k_0 n_0 m_0 l_0 + 2k_0 n_0(\mathbf{lm}) + 2m_0 l_0(\mathbf{nk}) + 2(\mathbf{nk})(\mathbf{lm})$$
$$-2\ k_0 m_0\ (\mathbf{nl}) - 2\ k_0 l_0\ (\mathbf{nm}) - 2\ n_0 m_0\ (\mathbf{kl}) - 2\ n_0 l_0\ (\mathbf{km})$$
$$-2\ (\mathbf{km})\ (\mathbf{nl}) + 2\ (\mathbf{kl})\ (\mathbf{nm})\ . \tag{101}$$

Indeed, equations (D.20) and (D.21) are the same.

However, the most simple form is (see (95):

$$G = \left| \begin{array}{cc} k_0 +\ \mathbf{k}\,\vec{\sigma} & n_0 -\ \mathbf{n}\,\vec{\sigma} \\ -l_0 -\ \mathbf{l}\,\vec{\sigma} & m_0 -\ \mathbf{m}\,\vec{\sigma} \end{array} \right| ,$$
$$\det G = (kk)\ (mm) + (nn)\ (ll) + 2\ [kn]\ [ml]$$
$$-2\ (\ k_0\ \mathbf{n} + n_0\ \mathbf{k} - i\ \mathbf{k}\times\mathbf{n}\ )\ (\ m_0\ \mathbf{l} + l_0\ \mathbf{m} + i\ \mathbf{m}\times\mathbf{l}\ )\ . \tag{102}$$

Let us consider several particular cases.

**Variant A**

$$+\mathbf{l} \to i\mathbf{l}\,, \qquad \mathbf{n} \to i\mathbf{n}\,, \qquad \mathbf{m} \to i\mathbf{m}\,, \qquad \mathbf{k} \to i\mathbf{k}\,,$$

$$G = \left| \begin{array}{cc} k_0 + \;\mathbf{k}\,\vec{\sigma} & n_0 - \;\mathbf{n}\,\vec{\sigma} \\ & \\ -l_0 - \;\mathbf{l}\,\vec{\sigma} & m_0 - \;\mathbf{m}\,\vec{\sigma} \end{array} \right| \to \left| \begin{array}{cc} k_0 + i\,\mathbf{k}\,\vec{\sigma} & n_0 - i\,\mathbf{n}\,\vec{\sigma} \\ & \\ -l_0 - i\,\mathbf{l}\,\vec{\sigma} & m_0 - i\,\mathbf{m}\,\vec{\sigma} \end{array} \right| ,$$

$$\det G = [kk]\,[mm] + [nn]\,[ll] + 2\,(kn)\,(ml)$$
$$+2\,(\,k_0\,\mathbf{n} + n_0\,\mathbf{k} + \mathbf{k}\times\mathbf{n}\,)\,(\,m_0\,\mathbf{l} + l_0\,\mathbf{m} - \mathbf{m}\times\mathbf{l}\,)\,. \qquad (103)$$

**Variant B**

$$m_a = k_a^*\,, \qquad l_a = n_a^*\,,$$

$$G(k,n) = \left| \begin{array}{cc} k_0 + \;\mathbf{k}\,\vec{\sigma} & n_0 - \;\mathbf{n}\,\vec{\sigma} \\ & \\ -n_0^* - \;\mathbf{n}^*\,\vec{\sigma} & k_0^* - \;\mathbf{k}^*\,\vec{\sigma} \end{array} \right| ,$$

$$\det G = (kk)\,(k^*k^*) + (nn)\,(n^*n^*) + 2\,[kn]\,[k^*n^*]\;-$$
$$-2\,(\,k_0\,\mathbf{n} + n_0\,\mathbf{k} - i\,\mathbf{k}\times\mathbf{n}\,)\,(\,k_0^*\,\mathbf{n}^* + n_0^*\,\mathbf{k}^* + i\,\mathbf{k}^*\times\mathbf{n}^*\,)\,. \qquad (104)$$

**Variant C**

$$m_0 = +k_0\,, \qquad l_0 = n_0\,, \qquad \mathbf{m} = -\mathbf{k}\,, \qquad \mathbf{l} = -\mathbf{n}\,,$$

$$G = \left| \begin{array}{cc} k_0 + i\,\mathbf{k}\,\vec{\sigma} & n_0 - i\,\mathbf{n}\,\vec{\sigma} \\ & \\ -n_0 + i\,\mathbf{n}\,\vec{\sigma} & k_0 + i\,\mathbf{k}\,\vec{\sigma} \end{array} \right|$$

$$\det G = [kk]^2 + [nn]^2 + 2\,(kn)^2\; - 2\,(\,k_0\,\mathbf{n} + n_0\,\mathbf{k} + \mathbf{k}\times\mathbf{n}\,)^2\,. \qquad (105)$$

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